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## FOUNDATIONS, THEORY OF SETS, LOGIC

**Skolem, Th.** The logical nature of arithmetic. *Synthese* 9 (1955), 375-384.

This is a philosophical essay. Skolem first compares the views of Dedekind, Poincaré, and certain empiricists. He then goes on to consider modern formalism. He points out that if we are to go beyond the mere verification of particular formulas, then an intuitive reasoning of some sort is necessary. This may either be platonistic, or of a sort naturally connected with definitions by induction.

H. B. Curry (University Park, Pa.).

**Manara, C. F.** La geometria nell'ambito del pensiero matematico. *Period. Mat.* (4) 34 (1956), 148-158.

**Freudenthal, Hans.** Axiom und Axiomatik. *Math.-Phys. Semesterber.* 5 (1956), 4-19.

An informal discussion with emphasis on the history of the subject.

**Trahtenbrot, B. A.** Definition of finite set and deductive incompleteness of the theory of sets. *Izv. Akad. Nauk SSSR. Ser. Mat.* 20 (1956), 569-582. (Russian)

Let  $\tau$  be any formal system of set theory, satisfying the following conditions: In  $\tau$  the axiom of extensionality is valid; in  $\tau$  the existence of the null set is provable; if  $a$  and  $b$  are given sets, then in  $\tau$  the existence of the following sets is provable:  $\{a\}$ ,  $a \cup b$ , the direct product of  $a$  and  $b$ , and, if  $a \in b$ , of  $b - a$ . Every formula  $\mathcal{A}$  of the first order predicate calculus  $P$  expresses, if interpreted, a condition for the cardinal number of the domain of individuals; this condition can be expressed in  $\tau$  by a formula  $\mathcal{A}(q)$ ; the transition of  $\mathcal{A}$  to  $\mathcal{A}(q)$  can be effected by a purely formal process. Let the superscript  $\Omega$ , if added to a formula  $\mathcal{A}$  of  $P$ , mean that  $\mathcal{A}$  is supposed to be identically true in every finite domain, but not in every infinite domain. The main result of the paper is: Given any formula  $\mathcal{A}$ , there exists a formula  $\mathcal{Q}^\Omega$  such that the formula (i):  $(q)(\mathcal{A}^\Omega(q) \rightarrow \mathcal{Q}^\Omega(q))$  is not provable in  $\tau$ . The proof utilizes the facts that the set  $K$  of the formulas  $\mathcal{Q}$ , for which  $(q)(\mathcal{A}^\Omega(q) \rightarrow \mathcal{Q}(q))$  is provable, is recursively enumerable, while the set  $K_\omega$  of finitely identical formulas is not, and  $KCK_\omega$ . By an analogous method the following theorem is proved: If  $\mathcal{Q}^\Omega(q)$  is not provable in  $\tau$ , then  $\mathcal{D}^\Omega$  exists, such that  $(q)(\mathcal{D}^\Omega(q) \rightarrow \mathcal{Q}^\Omega(q))$  is not provable in  $\tau$ . Finally, the following result is derived: If  $\tau$  is formally consistent, there exist in  $\tau$  undecidable formulas of the form (i). Proof. Let  $A$  and  $B$  be recursively enumerable sets of natural numbers, which are not recursively separable. Let  $\tilde{M} = k$  be an abbreviation for the formula in  $P$ , expressing that the predicate  $M$  holds for exactly  $k$  elements. Then there exists a formula  $\mathcal{A}$  such that  $K \in A$ , if and only if  $\mathcal{A} \& \tilde{M} = k$  is satisfiable in a finite domain [Trahtenbrot, *Dokl. Akad. Nauk SSSR (N.S.)* 70 (1950), 569-572, th. 1; *MR* 11, 488]. Let us abbreviate  $\sim(\mathcal{A} \& \tilde{M} = k)$  by  $\mathcal{A}_m$ , and  $(q)(\mathcal{A}_m(q) \rightarrow \mathcal{Q}_m(q))$  by  $S_m$ . Let  $S'$  be the set of the numbers  $m$ , for which  $S_m$  is provable in  $\tau$ , and  $S''$  that of the numbers  $m$  for which  $S_m$  is dis-

provable.  $S'$  and  $S''$  are recursively enumerable,  $ACS'$ ,  $BCS''$ . If  $S' \cup S''$  contained all natural numbers, then  $S'$  and  $S''$  would be recursive; let  $m \notin S' \cup S''$ , then  $S_m$  is undecidable in  $\tau$ . A. Heyting (Amsterdam).

**Hajnal, András; and Kalmár, László.** An elementary combinatorial theorem with an application to axiomatic set theory. *Publ. Math. Debrecen* 4 (1956), 431-449.

Some knowledge of Gödel's "The consistency of the continuum hypothesis" [Princeton, 1940; *MR* 2, 66] is desirable but not necessary. The sequence  $\langle x_1, \dots, x_n \rangle$  ( $n=2, 3, \dots$ ) is defined by induction: 1.  $\langle x_1, x_2 \rangle$  is an ordered pair  $(=\{\{x_1\}, \{x_1, x_2\}\})$ ; 2.  $\langle x_1, \dots, x_{n+1} \rangle = \langle x_1, \langle x_2, \dots, x_{n+1} \rangle \rangle$  ( $n=2, 3, \dots$ ). An elementary formula  $\Phi(x_1, \dots, x_n)$  (it is designed to replace the "definite Aussage" of Zermelo in his "Axiom der Aussonderung") is built up from propositions of the form  $y \in z$  and  $y = z$ , where instead of  $y$  and  $z$  any set- or class-variables may stand, by means of the logical operations and quantifiers, binding set-variables only. The central problem is to prove, as substitute for Zermelo's "Axiom der Aussonderung", for each elementary formula  $\Phi(x_1, \dots, x_n)$ , by means of certain axioms, the following theorem: (I) There is a class containing those and only those sequences of sets of the form  $\langle x_1, \dots, x_n \rangle$  for which  $\Phi(x_1, \dots, x_n)$  holds. In order to satisfy this requirement for each elementary formula  $\Phi(x_1, \dots, x_n)$  it is sufficient, on account of axioms A1-A3 and B1-B4 of Gödel (I) to satisfy it in the particular case when  $\Phi(x_1, \dots, x_n)$  is either of the form  $x_p \in x_q$  with  $1 \leq p, q \leq n$  and  $p \neq q$  or of the form  $x_p \in y$  with  $1 \leq p \leq n$ ,  $y$  being a class- or set-variable different from  $x_1, \dots, x_n$ , further to ensure the satisfaction of two conditions which are not relevant for this summary. This requirement (I) is, in its turn, satisfied if, besides the axiom B1 of Gödel, the following two propositions are available: 1. For any class  $A$ , there is a class  $B$  such that, for any sets  $x_1, \dots, x_n$  the sequence  $\langle x_1, \dots, x_n \rangle$  is contained in  $B$  if and only if  $x_p \in A$ ; 2. for any class  $A$  there is a class  $B$  such that, for any sets  $x_1, \dots, x_n$  the sequence  $\langle x_1, \dots, x_n \rangle$  is contained in  $B$  if and only if  $\langle x_p, x_q \rangle \in A$ . These propositions, however, cannot be used as axioms, for they represent an infinity of propositions, but they are replaceable by the axioms B5-B7 of Gödel. Therefore the axioms A1-A4 and B1-B7 of Gödel suffice to have the theorem (I) as a consequence. Actually, it turns out that the axiom B $^p$  of Gödel is redundant.

In the proof of the last result the authors use the following "combinatorial" theorem, which has also some independent significance: Given the primary operations 1.  $x \rightarrow \langle y, x \rangle$ , 2.  $\langle x, y \rangle \rightarrow \langle y, x \rangle$  and 3.  $\langle x, y, z \rangle \rightarrow \langle z, x, y \rangle$ , every operation of the form  $x_p \rightarrow \langle x_1, \dots, x_n \rangle$  and  $\langle x_p, x_q \rangle \rightarrow \langle x_1, \dots, x_n \rangle$ ,  $n=1, 2, \dots$ ;  $p, q=1, \dots, n$ ,  $p \neq q$ , can be obtained as a derived operation.

The representation distinguishes itself by all the qualities of a scientific representation, e.g. by lucidity, caution, simplicity and completeness. B. Germansky:

**Kurepa, G.** Still about induction principles. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke 302 (1955), 77-86. (Serbo-Croatian. English summary)

The author considers partially ordered sets  $S$ . A subset  $X$  of  $S$  is called a chain if any two elements of  $X$  are comparable. The segment  $[a, b]_S$  is the set of elements  $x$  which satisfy  $x=a$ , or  $x=b$ , or  $a < x$  and  $x < b$ . The author obtains equivalences of which the following is typical: (I) Each chain of  $S$  is finite and  $S$  is a segment. (I') If some segment of  $S$  is contained in the set  $M$ , and for every proper subsegment  $s$  of  $S$  which is contained in  $M$ , there is a segment  $s' \supset s$  of  $S$  also contained in  $M$ , then  $M \supseteq S$ .

G. Kreisel (Princeton, N.J.).

**Devidé, Vladimir.** Eine Charakterisierung des Ordnungstypus  $\omega + \omega$  der Menge der ganzen Zahlen mittels der Nachfolger-Funktion. Glasnik Mat.-Fiz. Astr. Drustvo Mat. Fiz. Hrvatske. Ser. II. 11 (1956), 11-15. (Serbo-Croatian summary)

This paper extends to the integers the author's remarks on Peano's axioms for the natural numbers [Arch. Math. 6 (1955), 408-412; MR 17, 448].

G. Kreisel.

**Noguera Barreneche, Rodrigo.** Demonstration of the general law of trichotomy. Stvdia. Rev. Univ. Atlantico I (1956), nos. 3-4-5, 151-171. (Spanish)

The author attempts to prove the trichotomy law for cardinal numbers and thus the axiom of choice by intuitive set theory. {The demonstration breaks down on page 154, lines 19-24, when, in asserting that  $Q$  has a certain maximal property, the axiom of choice is used}.

S. Ginsburg (Hawthorne, Calif.).

**Kemeny, John G.** A new approach to semantics. I. J. Symb. Logic 21 (1956), 1-27.

Two classes of semantic concepts are distinguished: those defined by Tarski, and the additional ones for which Carnap has offered definitions. These last definitions having met with criticism, the author sets out to construct new definitions for both classes. The object language  $L$  is, essentially, Church's formulation of the theory of types; in addition, there are a meta-language  $ML$  and a meta-meta-language  $MML$ .

The author's syntactic definition of a model is slightly modified; the intended models are called interpretations, and a well-formed formula (wff) of  $L$  is called analytically true if it is valid in all interpretations of  $L$ .

Then a certain interpretation  $M^*$  is singled out for the purpose of translating  $L$  into  $ML$ , and a wff of  $L$  is called true if it is valid in  $M^*$ . On the basis of this construction, numerous other concepts are introduced.

E. W. Beth (Baltimore, Md.).

**Kemeny, John G.** A new approach to semantics. II. J. Symb. Logic 21 (1956), 149-161.

As a formalised meta-language  $ML$  the author now uses Gödel's formalisation of set theory, dropping the axioms of replacement and choice and adding certain set-constants and meaning postulates. Then a formalisation of his construction [see the preceding review] is carried out in detail.

Now the meta-meta-language  $MML$  is used in a discussion of the relation of  $L$  and  $ML$ .  $MML$  is made to serve as a meta-language for  $L$  as well by singling out, for the purpose of translating  $L$  into  $MML$ , the interpretation  $M^*$  of  $L$  (obtained by translating  $M^*$  into  $MML$ ).

Thus a basis is obtained for defining Quine's concepts of a relation in the Theory of Reference and of a relation in the Theory of Meaning. Both kinds of relations are called semantic relations. Following a suggestion made by Church, the author proves that semantic relations are invariant under free translation.

E. W. Beth.

**Fuentes Miras, José Ramón.** "Deductive truth" according to Tarski. Gac. Mat., Madrid (I) 6 (1954), 110-120. (Spanish)

This is a summary of Tarski's famous paper on the Wahrheitsbegriff [Studia Philos. 1 (1935), 261-405]. The discussion pertains principally to §§ 1-3 of Tarski's monograph, with very brief mention, at the very end, of the rest.

H. B. Curry (University Park, Pa.).

**Svenonius, Lars.** Definability and simplicity. J. Symb. Logic 20 (1955), 235-250.

A study is made of how a definition of a relation  $R$  in terms of a relation  $R'$  works in a given universe  $U$ , the definition being regarded as a correlation between the "possible extensions" of  $R$  and  $R'$ . Specifically, the author is concerned with such questions as: "is every relation of the form  $s$  definable in terms of some relation of the form  $s'$ ?" and with Goodman's complexity criterion. A peculiar terminology is used.

The extension of a relation  $R$  in a universe  $U$  is represented by an ordered couple  $(R, u)$ , where  $u$  is a matrix in  $U$ . An extension of a class  $K$  of relations correlates with each  $R \in K$  a certain extension  $(R, u)$ . Two extensions of a class  $K$  are called isomorphic if one is obtained from the other by a permutation  $P$  of the universe  $U$ .  $K$  is said to be homogeneous if it is closed under isomorphy. To each class  $K$  a class  $t(K)$  of possible extensions is assigned in accordance with three special postulates.

Besides classes  $K$ , the author considers (finite)  $r$ -sequences  $S$  of relations  $R$  and he assigns to each  $r$ -sequence  $S$  a class  $m(S)$  of possible extensions, which is called the form of  $S$ .

If  $K \subseteq K'$ , then clearly each extension in  $t(K')$  determines a unique extension in  $t(K)$ .

Now two extensions of arbitrary classes  $K$  and  $K'$  are called compatible if  $t(K'')$  [where  $K''$  is the sum of  $K$  and  $K'$ ] contains an extension by which they are both determined. And if each extension in  $t(K)$  is compatible with exactly one extension in  $t(K')$ , then  $K'$  is said to be definable in terms of  $K$ ; the mapping which, with each extension in  $t(K)$ , connects the compatible extension in  $t(K')$  is called the definition mapping of  $t(K)$  onto  $t(K')$ .

Goodman's criterion for complexity can now be re-stated as follows: let  $m$  and  $m'$  be any forms. Then  $\text{complexity}(m) \leq \text{complexity}(m')$  whenever, to every  $r$ -sequence  $S$  with  $m(S) \subseteq m$  there is an  $r$ -sequence  $S'$  with  $m(S') \subseteq m'$ , such that  $S$  and  $S'$  are mutually definable.

E. W. Beth (Baltimore, Md.).

**Ackermann, Wilhelm.** Begründung einer strengen Implikation. J. Symb. Logic 21 (1956), 113-128.

The present paper is motivated by the desire to present a propositional connective which can be interpreted as a consequence-relation more truly than the usual (material) implication of the calculus of propositions. To this extent, the author follows the line of thought of C. I. Lewis but the "strong" (streng) implication introduced here (symbol  $\rightarrow$ ) is distinct from the strict implication of C. I. Lewis. In particular, the present author does not accept the general validity of  $B \rightarrow (A \rightarrow A)$  since the validity of

$A \rightarrow A$  is quite independent of the validity of  $B$ . Similarly, he rejects the validity of  $(A \wedge \bar{A}) \rightarrow B$ , while accepting  $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ .

As a first step, a new system  $\Sigma$  of the classical propositional calculus is presented which is, roughly speaking, midway between an ordinary axiomatic calculus and Gentzen's calculus of natural deduction.  $\Sigma$  is shown to be equivalent to the standard calculus of propositions, more particularly to the system of Hilbert and Bernays. It is then modified (chiefly as regards its rules of deduction) so as to yield a system  $\Sigma'$  in which the implication is in keeping with the general ideas indicated earlier.  $\Sigma'$  in turn is shown to be equivalent to another system  $\Pi'$  which is related more closely to the system of Hilbert and Bernays.  $\Pi'$  can be extended so as to constitute a lower predicate calculus.

Finally, the author discusses some concepts of Gödel Logic, and shows that the notions of necessity and of impossibility can be formulated within his system, if one first introduces a new symbol which indicates "Absurdity". {It appears to the reviewer that the introduction of such a symbol is not quite in keeping with the general spirit of the paper.} *A. Robinson.*

**Rogers, Hartley, Jr. Certain logical reduction and decision problems.** *Ann. of Math.* (2) **64** (1956), 264-284.

The principal purpose of this paper is the reduction of the problem of the (universal) validity of a formula  $A$  in the lower predicate calculus to the validity of a related formula  $A'$  in a particular class of mathematical structures,  $C$ . Using a well-known result of Kalmar's, it is sufficient to consider only formulae  $A$  which involve but a single-binary-relation. The author then shows how to obtain a formula  $A'$  such that  $A$  is (universally) valid if and only if  $A'$  holds in all partially ordered structures; also, how to obtain an  $A'$  which includes two binary relations such that  $A$  is valid if and only if  $A'$  holds in all structures in which the relations of  $A'$  can be interpreted as equivalence relations; or again, how to define  $A'$  such that  $A$  is valid if and only if  $A'$  holds in all lattices; or such that  $A'$  (containing a single ternary relation) holds in all semi-groups if and only if  $A$  is valid. Numerous interesting corollaries are included. The transformations from  $A$  to  $A'$  are given effectively. It follows that the non-existence of a decision procedure for the lower predicate calculus implies the non-existence of decision procedures for the classes of structures,  $C$ , which arise in the present analysis. *A. Robinson* (Toronto, Ont.).

**Jaśkowski, S. Undecidability of first order sentences in the theory of free groupoids.** *Fund. Math.* **43** (1956), 36-45.

A groupoid is an algebraic structure consisting of a set  $F$  and an operation  $\gamma$  with the following property: There exists a subset  $G$  of  $F$  such that every element  $a$  of  $F$  is uniquely represented in the form  $a = \theta_\gamma(g_1 \cdots g_n)$ , where  $g_i \in G$  and  $\theta_\gamma(g_1 \cdots g_n)$  is a term made up of the symbols  $g_1, \dots, g_n$  and the symbol  $\gamma$ . The author gives an effective method for associating with any formula  $A$  of the predicate calculus of first order, a formula  $A'$  whose non-logical symbols are a monadic predicate symbol  $E$  and a function symbol  $\gamma$ , and shows:  $A$  is satisfiable if and only if  $A'$  is satisfiable by a groupoid  $(F_0, \gamma_0)$  and a subset  $E_0$  of  $F_0$ . The result shows (by the undecidability of the predicate calculus) that the elementary theory of groupoids is undecidable, and leads to a short proof of theorem 35

of Pepis, *Fund. Math.* **30** (1938), 257-348. The paper is another contribution to the general reduction problem for the predicate calculus which has recently been discussed by Rogers [see the paper reviewed above].

*G. Kreisel* (Princeton, N.J.).

**Maehara, Shôji. Gentzen's theorem on an extended predicate calculus.** *Proc. Japan Acad.* **30** (1954), 923-926.

The author outlines a proof of the principal theorem (Hauptsatz, also called elimination theorem) of Gentzen's thesis [*Math. Z.* **39** (1934), 176-210, 405-431] for that extension of the classical predicate calculus (Gentzen's LK) which is formed by admitting quantification of propositional variables. *H. B. Curry* (University Park, Pa.).

**Maehara, Shôji. The predicate calculus with  $\epsilon$ -symbol.** *J. Math. Soc. Japan* **7** (1955), 323-344.

Verf. gibt mithilfe des Gentzenschen Hauptsatzes einen rein-syntaktischen Beweis des (semantisch leicht zu erhaltenden) Satzes: Ist ein Axiomensystem im elementaren Prädikatenkalkül wf. so auch nach Erweiterung des Prädikatenkalküls durch das Hilbertsche  $\epsilon$ -Symbol mit der Regel  $F(a) \rightarrow F(\epsilon_x F(x))$ . Als wichtigstes Hilfsmittel wird gebraucht: Ist aus einem wf. Axiomensystem  $A$  die Aussage  $\forall x_1 \cdots x_m \exists y \mathcal{A}(y; x_1 \cdots x_m)$  ableitbar, so ist das System  $A'$ , das aus  $A$  durch Erweiterung mit

$$\forall x_1 \cdots x_m \mathcal{A}(f(x_1, \dots, x_m); x_1, \dots, x_m)$$

entsteht, ebenfalls wf. [Hier ist  $f$  ein Funktionssymbol, das sonst nicht vorkommt.] Verfasser gibt auch die Verallgemeinerung des Satzes für den Prädikatenkalkül mit Identität und für den vom Verfasser eingeführten Prädikatenkalkül mit gebundenen Aussagenvariablen [siehe das vorstehende Referat]. *P. Lorenzen* (Bonn).

**Takeuti, Gaisi. Construction of ramified real numbers.** *Ann. Japan Assoc. Philos. Sci.* **1** (1956), 41-61.

The author describes a subsystem RS of his system GLC [*Jap. J. Math.* **23** (1953), 39-96; **24** (1954), 149-156; *MR* **17**, 701]. RS is a formalization of the ramified theory of (finite) types without extensionality. RS satisfies the cut theorem of Gentzen [*Math. Z.* **39** (1934), 176-210, 405-431]. RS' is a system based on RS in which the variables of lowest type may be interpreted to range over the rationals; the equality axioms are restricted to rational operations and predicates; RS' contains a predicate  $n(a)$ , meaning that  $a$  is a natural number, and the schema of induction for elements satisfying  $n(a)$ . The consistency of RS' is proved by a modification of Gentzen's consistency proof for arithmetic [*Math. Ann.* **112** (1936), 493-565]. Finally a system RR of ramified analysis is described, which is based on RS', and stated to be consistent. {The system seems closely related to one discussed by Schütte [*Math. Ann.* **124** (1952), 123-147; *MR* **13**, 615] which the author does not mention.}

*G. Kreisel* (Princeton, N.J.).

**Rose, Alan. Formalisation du calcul propositionnel implicatif à  $\aleph_0$  valeurs de Łukasiewicz.** *C. R. Acad. Sci. Paris* **243** (1956), 1183-1185.

In a forthcoming paper, Rosser and the author show that, in confirmation of a conjecture of Łukasiewicz [cf. Łukasiewicz and Tarski, *C.R. Soc. Sci. Lett. Varsovie. Cl. III.* **23** (1930), 30-50], the  $\aleph_0$ -valued propositional calculus of Łukasiewicz can be axiomatized by the

following set of five formulas.

- A1.  $P \rightarrow (Q \rightarrow P),$   
 A2.  $(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R)),$   
 A3.  $(P \vee Q) \rightarrow (Q \vee P),$   
 A4.  $(P \rightarrow Q) \vee (Q \rightarrow P),$   
 A5.  $(\bar{P} \rightarrow \bar{Q}) \rightarrow (Q \rightarrow P).$

The alternation  $A \vee B$  stands for  $(A \rightarrow B) \rightarrow B$ .

In the paper under review, the author deduces from the above result that A1-A4 form an axiom system for the Łukasiewicz  $\aleph_0$ -valued implicational propositional calculus.

In the lemma on p. 1184, the author seems to have proved the stronger assertion that the formula  $Q$  may be taken to be a propositional variable. Also, on line 7 of p. 1184, " $\bar{P}$ " should be replaced by " $P$ ". *E. Mendelson.*

**Medvedev, Yu. T. On nonisomorphic recursively enumerable sets.** Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 211-214. (Russian)

This paper is based on Post's paper [Bull. Amer. Math. Soc. 50 (1944), 284-316; MR 6, 29] on recursively enumerable (r.e.) sets of positive rational integers (p.i.). Two such sets are called isomorphic when either can be carried into the other by a one-one recursive transformation. The paper deals with the question of the existence of nonisomorphic types of hypersimple sets (as defined by Post). Given two sets  $E'$  and  $E''$  of p.i.,  $E'$  is said to be not less dense than  $E''$ . If there is a recursive function  $f(n)$  such that the number of p.i.  $\leq f(n)$  in  $E'$  is not less than the number of those  $\leq n$  in  $E''$ ,  $E'$  is said to be more dense than  $E''$  if this relation holds but not the converse. In the latter case  $E'$  and  $E''$  are nonisomorphic. It is shown that a hypersimple set is a r.e. set with an infinite complement which is less dense than the set of all p.i.; also that, given a hypersimple set  $H_0$ , then by successively taking away single p.i. from  $H_0$  one can form a sequence  $H_0, H_1, \dots$  of hypersimple sets whose complements are of constantly increasing density. (The converse process is also possible, but is nonconstructive.) It is stated without proof that an analogous process can be continued into the constructive transfinite. {In the first formula in the proof of Theorem 3 the symbols  $\theta$  and  $\varphi$  appear to have been interchanged.} *H. B. Curry.*

**Gentzen, Gerhard. Zusammenfassung von mehreren vollständigen Induktionen zu einer einzigen.** Arch. Math. Logik Grundlagenforsch. 2 (1954), 1-3.

This posthumous note was written in honor of the 60th birthday of H. Scholz (Dec. 17, 1944). It shows that in any reasonable formulation of arithmetic, admitting the processes of the first order predicate calculus with equality, a proof involving any number applications of mathematical induction can be transformed into one involving only one such application. *H. B. Curry* (University Park, Pa.).

**Robinson, Raphael M. Arithmetical representation of recursively enumerable sets.** J. Symb. Logic 21 (1956), 162-186.

Nach M. Davis [J. Symb. Logic 18 (1953), 33-41; MR 14, 1052] gibt es ein  $\lambda$ , sodass sich jede (rekursiv) aufzählbare Menge  $S$  natürlicher Zahlen mit einem geeigneten  $\lambda+3$ -stelligen ganzzahligen Polynom  $P$  darstellen lässt durch

$$y \in S \leftrightarrow \forall b \wedge w [w \leq b \rightarrow \forall x_1, \dots, x_\lambda P(y, b, w, x_1, \dots, x_\lambda) = 0].$$

Verfasser beweist auf einem neuen ("somewhat intricate") Wege, dass  $\lambda=4$  gewählt werden kann.  $\lambda=0$  ist unmöglich,  $0 < \lambda < 4$  bleibt offen. Der Beweis ergibt das schärfere Resultat, dass es ein 8-stelliges ganzzahliges Polynom  $Q$  gibt, so dass sich jede (rekursiv) aufzählbare Menge  $S$  natürlicher Zahlen mit einem geeigneten  $n$  darstellen lässt durch

$$y \in S \leftrightarrow \forall b \wedge w [w \leq b \rightarrow \forall x_1, \dots, x_4 [x_1 \leq b \wedge x_2 \leq b \wedge x_3 \leq b \wedge x_4 \leq b \wedge Q(n, y, b, w, x_1, \dots, x_4) = 0]].$$

*P. Lorenzen* (Bonn).

★ **Henkin, L. La structure algébrique des théories mathématiques.** Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1956. 53 pp. 900 francs.

This book is an elaboration of lectures given by the author in Paris in 1955. It is very well written and can be read with little preliminary knowledge of the subject.

The first chapter contains an introduction to the theory of Boolean algebras, culminating in a proof of Stone's representation theorem. In the second chapter the author discusses the now familiar relation between deductive systems (viewed here from the point of view of Boolean algebra) and the structures, or models, which are described by such systems. The effective application of these notions to Algebra is a subject of which the present author was a pioneer and some simple examples are given here. An appendix which pertains to this chapter contains a derivation of the extended completeness theorem of the lower predicate calculus from Stone's representation theorem. {The reviewer cannot agree with a remark at the bottom of p. 47 which seems to imply that for a countable language completeness can be proved without the use of the axiom of choice (or at least of the maximal ideal theorem).} Concluding the chapter, there is a discussion of the modifications required in the definition of a Boolean algebra to conform to the intuitionist formulation of the propositional calculus.

The last chapter of the book constitutes its most original part. In it the author discusses an axiomatic theory (the theory of cylindrical algebras) which is related to the lower predicate calculus as the theory of Boolean algebras is related to the calculus of propositions. A representation theorem is established under suitable conditions. The author's ideas on the subject are related to those of Tarski. {It would be interesting to have a detailed comparison of the present theory with the theory of Polyadic algebras of P. R. Halmos, which has the same purpose.}

The need for the following corrections has been pointed out by the author: p. 39, l. 14, change " $y \in G$ " to " $y \notin G$ "; l. 22, change " $\varphi$ " to " $\epsilon$ "; l. 25, change " $x$ ," to " $x_1$ ".

*A. Robinson* (Toronto, Ont.).

**Sestakov, V. I. Modelling the operations of the propositional calculus by means of the simplest four-pole networks.** Vyčisl. Mat. Vyčisl. Tehn. 1 (1953), 56-89. (Russian)

The author discusses various ways of representing operations of propositional algebra by combinations of circuit units. Three kinds of such units are considered, viz. two-pole units, four-pole units, and commutators. A two-pole unit is simply an admittance between two terminals; for the representation considered the admittance is either 0 (open circuit) or  $\infty$  (short-circuit). A four-pole unit is a device with two input and two output

terminals; the representation can be in terms of either the voltage ratio or the current ratio of output to input; the situation lends itself to representation in terms of 0 and 1. A commutator is a four-pole unit such that the output is either the same as the input or is the same with reversed polarity; this lends itself to identification of truth and falsity with 1 and -1. The paper discusses the basic logical connections and a suitable algebraic expression in all these cases. At the end there is a discussion of the

design of a binary adder. (The treatment is elementary and clear; its newness the reviewer is not able to judge.)  
H. B. Curry (University Park, Pa.).

See also: Massera, p. 227; Pacheco de Amorin, p. 240; Vietoris, p. 240; Rohleder, p. 258; Šik, p. 192; Burks and Copi, p. 239; Froda, p. 274; Jakubik, p. 275; Scott, p. 328; Beth and Tarski, p. 328; Tarski, p. 328.

## ALGEBRA

### Combinatorial Analysis

Aigner, Alexander. Über die systematische Lösung einer Wägungsaufgabe. Math.-Phys. Semesterber. 5 (1956), 162-163.

Given  $\frac{1}{2}(3^n - 1)$  coins of identical appearance, at most one of which has an incorrect weight, and an extra coin which is known to have the correct weight, it is required to identify the defective coin (if it exists) and to decide whether it is too heavy or too light, using a balance (without weights)  $n$  times. The author solves this problem, giving full details for the case when  $n=3$ . [For the "last word" on the subject, see C. A. B. Smith, Math. Gaz. 31 (1947), 31-39.] H. S. M. Coxeter (Toronto, Ont.).

See also: Ford and Uhlenbeck, p. 326.

### Linear Algebra

Ballieu, Robert. Produits scalaires à annulation symétrique. Ann. Soc. Sci. Bruxelles. Sér. I. 70 (1956), 87-95.

Let  $K$  be a division ring,  $w$  an antiautomorphism of  $K$ ,  $L$  a left vector space over  $K$  and  $f$  a scalar product on  $L$  relative to  $w$ , for which  $f(x, y) = 0$  implies  $f(y, x) = 0$ . Then the main result is that, unless the  $x$  with  $f(x, y) = 0$  for all  $y$  in  $L$  form a subspace of codimension 0 or 1 in  $L$ , there is a scalar  $\delta$  in  $K$  with (1)  $w(f(x, y)) = \delta f(y, x)$ , (2)  $w^2(\lambda) = \delta \lambda \delta^{-1}$  and (3)  $\delta w(\delta) = 1$ . Conversely, given  $\delta$  satisfying (2) and (3), an  $f$  can be constructed satisfying (1).

A. P. Robertson (Glasgow).

Roelcke, W. Über die Verteilung der Klassen eigentlich assoziierter zweireihiger Matrizen, die sich durch eine positiv-definite Matrix darstellen lassen. Math. Ann. 131 (1956), 260-277.

Let  $Q$  be a  $m$ -rowed positive definite symmetric matrix with integer elements and with even diagonal element. Let  $a(T)$  be the number of  $m \times 2$  matrices  $G$  with integer elements such that  $G'QG = T$ , where  $T = \begin{pmatrix} t_0 & t_1 \\ t_1 & t_2 \end{pmatrix}$  is a two-rowed integer matrix with even  $t_0$  and  $t_2$  satisfying (\*)  $|2t_1| \leq t_0 \leq t_2$ . Let  $\varepsilon(T)$  be the number of unimodular matrix  $U$  such that  $U' + U = T$ . Let

$$T = u \begin{pmatrix} (x^2 + y^2)y^{-1} & xy^{-1} \\ xy^{-1} & y^{-1} \end{pmatrix}$$

with  $u = (\det T)^{\frac{1}{2}}$ . The domain (\*) is equivalent to  $\mathfrak{F}$ :  $|2x| \leq 1$ ,  $x^2 + y^2 \geq 1$ ,  $y > 0$ . Let  $\mathfrak{G}$  be subset of  $\mathfrak{F}$ . Then

$$\sum \frac{a(T)}{\varepsilon(T)} \sim \frac{3}{\pi} \text{area}(\mathfrak{G}) (4\pi)^m / (12m\Gamma(m-1)),$$

where the sum runs through  $T$  with  $u \leq q$  and  $(x, y) \in \mathfrak{G}$ .

The proof follows in principle the line of Hecke [Math. Z. 1 (1918), 357-376; 6 (1920), 11-51] with the aid of the properties of modular functions of the second degree.

L. K. Hua (Peking).

Marathe, C. R. A note on quasi-idempotent matrices. Amer. Math. Monthly 63 (1956), 632-635.

The author studies the elementary properties of quasi-idempotent matrices [Huff, same Monthly 62 (1955), 334-339; MR 16, 989]. For example, he shows that if  $G(x)$  is the associated exponential polynomial matrix of a quasi-idempotent  $H$ , and  $G(0)$  commutes with a quasi-idempotent  $A$ , then  $H^{-1}AH$  is also quasi-idempotent, where  $H^{-1} = G(-1)$ . G. B. Huff (Athens, Ga.).

Ingleton, A. W. The rank of circulant matrices. J. London Math. Soc. 31 (1956), 445-460.

The  $n \times n$  circulant matrix  $(c_{ij})$  ( $c_{ij} = a_{i-j}$ ) is called non-recurrent if the period of the sequence  $\{a_k\}$  is exactly  $n$ . The problem is to choose  $c_{ij}$  according to these conditions ( $n$  is given) such that the rank of the matrix is minimal. The rank of  $(c_{ij})$  equals  $\sum_{r \in R} \varphi(r)$ , where  $\varphi$  is Euler's function, and  $R$  is the set of those divisors of  $n$  for which the  $r$ th cyclotomic polynomial does not divide  $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ . The matrix is non-recurrent if the numbers of the set  $R$  have  $n$  as their l.c.m. If  $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$ , the minimal rank turns out to be  $\sum_{i=1}^m p_i^{\alpha_i} - 1 - \eta(n)$ , where  $\eta(n) = 1$  if  $n \equiv 2 \pmod{4}$ ,  $n > 2$ , and  $\eta(n) = 0$  otherwise. Restriction of the matrix elements to integers does not change the minimum, but restriction to non-negative integers increases the minimum by 1. The larger part of the paper is devoted to the case that the matrix elements are restricted to the values 0 and 1. The minimum is determined only for a special class of values of  $n$ ; in the general case upper and lower bounds for the minimum are given.  
N. G. de Bruijn (Amsterdam).

Marcus, M.; and McGregor, J. L. Extremal properties of Hermitian matrices. Canad. J. Math. 8 (1956), 524-531.

Let  $A$  be an  $n \times n$  positive definite Hermitian matrix, with eigenvalues  $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ . Let  $Q_{kr}$  denote the  $\binom{k}{r}$  sets of integers  $(i_1, i_2, \dots, i_r)$  such that  $1 \leq i_1 < i_2 < \dots < i_r \leq k$ , where  $1 \leq k \leq n$ ; for any  $\omega \in Q_{kr}$ , let  $x_\omega$  be the Grassmann product  $x_{i_1} \wedge \dots \wedge x_{i_r}$ . Write

$$E_r(\alpha_1, \dots, \alpha_k) = \sum_{\omega \in Q_{kr}} \prod_{j=1}^r \alpha_{i_j}.$$

The main result of the paper is as follows. If  $C_r(A)$  is the  $r$ th compound of  $A$ , then

$$\min_{\omega \in Q_{kr}} \sum (C_r(A)x_\omega, x_\omega) = E_r(\alpha_1, \dots, \alpha_k),$$

$$\max_{\omega \in Q_{kr}} \sum (C_r(A)x_\omega, x_\omega) = E_r(\alpha_n, \dots, \alpha_{n-k+1}),$$

the extrema being taken over all orthonormal sets  $x_1, x_2, \dots, x_k$  of  $k$  vectors. This reduces to Ky Fan's theorem [Proc. Nat. Acad. Sci. U.S.A. 35 (1949), 652-655; MR 11, 600] in the special case  $r=1$ . Numerous other inequalities are proved, including some relations between the eigenvalues of  $A$ ,  $B$  and  $A+B$  when  $A$  and  $B$  are non-negative definite Hermitian matrices. *F. Smithies.*

**Sarkar, Sib Sankar.** On a matrix representation of homogeneous algebraic forms. Bull. Calcutta Math. Soc. 47 (1955), 227-230.

The author represents homogeneous algebraic forms  $\sum a_{i_1 \dots i_p} x_{i_1} \dots x_{i_p}$  ( $i_j=1, \dots, n$  for  $j=1, \dots, p$ ) by means of  $p$ -way matrices, according to the Cayley-Rice law of multiplication. For example, if  $n=p=3$  and  $S$  is the 3-way symmetric matrix  $\|a_{i_1 i_2 i_3}\|$ , then the form is  $X(XS)X^t = X(SX)X$ . *N. G. de Bruijn (Amsterdam).*

**Nalli, Pia.** Calcolo tensoriale ed operazioni funzionali. Boll. Un. Mat. Ital. (3) 11 (1956), 117-122.

This paper is a continuation of same Boll. (3) 10 (1955), 135-146 [MR 17, 407]. From the  $n^3$  components of a tensor  $T_i^{j,k}$  a set of  $n^2-1$  linear homogeneous functions are considered that transform linearly in themselves. It is a mistake that these functions should not be components of a tensor, in fact they are components (though a bit unusual ones) of the tensor  $T_i^{j,k} - n^{-1} T_j^i A_k^j$  (cf. referent's remarks on the previous paper). Sometimes it may be convenient to choose components of a tensor in such a queer way and this is illustrated in the second part of the paper where an application to integral equations is given. *J. A. Schouten (Epe).*

**Wilansky, Albert.** The row-sums of the inverse matrix. II. Amer. Math. Monthly 63 (1956), 652-653.

A continuation of the article listed in MR 13, 311.

See also: El Makarem, p. 301; Erugin, p. 307; Bauer, p. 308; van der Woude, p. 329; Dupac, p. 336; Reiersøl, p. 344; Searle, p. 346; Ivlev, p. 352.

### Polynomials

**Carlitz, L.** Solvability of certain equations in a finite field. Quart. J. Math. Oxford Ser. (2) 7 (1956), 3-4.

Let  $q=p^n$ , where  $p$  is a prime, and let  $\Delta$  denote the finite field of order  $q$ . Let  $g(x_1, x_2, \dots, x_k)$  denote an arbitrary polynomial with coefficients in  $\Delta$  in  $k$  independent indeterminates  $x_1, x_2, \dots, x_k$ . The author proves Theorem 1: Let  $k|p-1$ , and let  $a_1, a_2, \dots, a_k$  be non-zero elements in  $\Delta$ , and let  $g(x_1, x_2, \dots, x_k)$  have degree less than  $k$ . Then, the equation  $\sum_{i=1}^k a_i x_i^k = g(x_1, \dots, x_k)$  has at least one solution in  $\Delta$ . As the author points out, the method of proof, which uses only the most elementary properties of finite fields, is the same as that used by St. Schwartz [same Quart. 19 (1948), 160-163; MR 10, 101] to prove Theorem 1 in the case where  $g(x_1, x_2, \dots, x_k)$  is constant. The author of the article under review proceeds to employ this method again to obtain an extension of Theorem 1: Let  $g(x_1, x_2, \dots, x_k)$  have degree less than  $k$ , and let  $f(x_1, x_2, \dots, x_k)=0$  be a homogeneous equation of degree  $k$  having  $m$  solutions in  $\Delta$ , where  $p$  does not divide  $m$ . Then  $f(x_1, x_2, \dots, x_k)=g(x_1, x_2, \dots, x_k)$  has at least one solution in  $\Delta$ . *C. C. Faith (East Lansing, Mich.).*

**Dürbaum, Hansjürgen.** Note on a paper by Satô. Bull. Earthquake Res. Inst. Tokyo 34 (1956), 19-20. (Japanese summary)

Elementary proof of a statement by Satô [same Bull. 28 (1950), 23-29; MR 13, 184] concerning the real root of a particular cubic equation. *E. Kogbetliantz.*

See also: Skopin, p. 276; Carlitz, p. 285; Klein, p. 329; Nunziante-Cesaro, p. 368.

### Continued Fractions

**Khinchine, A. [Hinčin, A. Ya.] Kettenbrüche.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1956. vi+96 pp. DM 5.40.

Translation into German by Viktor Ziegler of the second Russian edition. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949 [MR 13, 444].

**Singh, Vikramaditya; and Thron, W. J.** On the number of singular points, located on the unit circle, of certain functions represented by  $C$ -fractions. Pacific J. Math. 6 (1956), 135-143.

Suppose the  $C$ -fraction  $1+K_1^\infty(d_n z^{a_n}/1)$  ( $d_n$  a complex number distinct from 0,  $a_n$  a positive integer) satisfies the following conditions: (1)  $\lim_{n \rightarrow \infty} (4/|d_n|)^{1/a_n} = 1$ , (2) there is an increasing sequence  $\{n_k\}$  of positive integers such that  $\lim_{k \rightarrow \infty} a_{n_k} = \infty$  and  $\lim_{k \rightarrow \infty} [n_k/k] < 2$  and (3)  $\liminf_{n \rightarrow \infty} [p_n/(h_n - p_n)] = t < 1$ , where  $h_n = \sum_{i=1}^n a_i$  and  $p_n$  is the maximum of the degrees of the  $n$ th numerator and denominator of  $1+K_1^\infty(d_n z^{a_n}/1)$ . It is shown that the  $C$ -fraction represents a function having a singular point not a pole on each arc of the unit circle of length greater than  $2\theta$ , where  $0 \leq \theta < \pi$ ,  $\cos \theta = 1 - 2\{t + [t(1+2)]^{1/2}\}/(1+2t)$ , if  $0 \leq t \leq \frac{1}{2}$ , and  $\cos \theta = -t$ , if  $\frac{1}{2} < t < 1$ . *H. S. Wall.*

See also: Le Veque, p. 283; Loewner, p. 318; Michalup, p. 336; Macon and Baskerville, p. 337.

### Partial Order Structures

**Froda, Alexandru.** Sur les réunions ordonnées d'ensembles. Acad. R. P. Române. Stud. Cerc. Mat. 7 (1956), 7-35. (Romanian. Russian and French summaries)

Let  $R = \bigcup \{A_i : i \in I\}$ , where the sets  $A_i$  are (completely) ordered and where  $R$  is ordered. Suppose that the order on the  $A_i$  induced by  $R$  coincides with the original order of the  $A_i$ . The author introduces and studies a number of properties of such sets, with special emphasis on well ordering. He then uses these results to study the arithmetic of infinite ordinal numbers. *E. Hewitt.*

**Kalman, J. A.** An identity for  $l$ -groups. Proc. Amer. Math. Soc. 7 (1956), 931-932.

It has been shown by Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1940; MR 1, 325] that

$$|a \vee c - b \vee c| + |a \wedge c - b \wedge c| = |a - b|$$

holds for all  $a, b$  and  $c$  in any vector lattice. This result is now shown to hold in any  $l$ -group. *J. Hartmanis.*

\* Lesieur, L. Treillis géométriques. I. Séminaire A. Châtelet et P. Dubreil de la Faculté des Sciences de Paris, 1953/1954. Algèbre et théorie des nombres. 2e tirage multigraphié, pp. 1-01 - 1-10. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956.

\* Lesieur, L. Treillis géométriques. II. Séminaire A. Châtelet et P. Dubreil de la Faculté des Sciences de Paris, 1953/1954. Algèbre et théorie des nombres. 2e tirage multigraphié, pp. 2-01 - 2-08. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956.

The author extends the theory of matroid lattices, as founded by H. Whitney [Amer. J. Math. 57 (1935), 509-533] and the reviewer [ibid. 57 (1935), 800-804], and deepened by MacLane, to the atomic, infinite-dimensional case. The ideas are closely related to those introduced by Frink [Trans. Amer. Math. Soc. 60 (1946), 452-467; MR 8, 309] to treat the modular case. Many essentially known results are given elegant new formulations.

G. Birkhoff (Cambridge, Mass.).

Skolem, Th. The abundance of arithmetic functions satisfying some simple functional equations. Norske Vid. Selsk. Forh., Trondheim 29 (1956), 47-53.

The author obtains various natural examples of "ringoids"  $R$ , possessing more than one addition and a multiplication distributive on all additions. He shows that, if  $ax=ay$  implies  $x=y$  for some  $a \in R$ , then multiplication cannot be two-sided distributive on more than one cancellative addition.

G. Birkhoff.

Mikulík, Miloslav. Beitrag zu topologischen Verbänden. Acta Acad. Sci. Czechoslovenicae Basis Brunensis 27 (1955), 368-372. (Czech. Russian and German summaries)

Let  $S$  be a metric lattice, with metric  $\rho$ , satisfying the following conditions. (A)  $\rho(x, y) = \rho(x \wedge y, x \vee y)$ . (B) If  $x < y < z$ , then  $\rho(x, y) < \rho(x, z)$ ,  $\rho(y, z) < \rho(x, z)$ . (C) Every monotone non-increasing (non-decreasing) sequence of points contains a convergent subsequence. For such metric lattices the author proves the continuity of operations  $\vee$  and  $\wedge$ , i.e. if  $x_n \rightarrow x$ ,  $y_n \rightarrow y$ , then  $x_n \vee y_n \rightarrow x \vee y$  and  $x_n \wedge y_n \rightarrow x \wedge y$  (Th. 1). If moreover  $S$  is complete, then  $S$  is a topological lattice in its order topology [for terminology, see Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, especially p. 82, Th. 16; MR 10, 673].

D. Kurepa (Zagreb).

\* Lesieur, L. Les treillis en topologie. I. Séminaire A. Châtelet et P. Dubreil de la Faculté des Sciences de Paris, 1953/1954. Algèbre et théorie des nombres. 2e tirage multigraphié, pp. 3-01 - 3-11. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956.

\* Lesieur, L. Les treillis en topologie. II. Séminaire A. Châtelet et P. Dubreil de la Faculté des Sciences de Paris, 1953/1954. Algèbre et théorie des nombres. 2e tirage multigraphié, pp. 4-01 - 4-10. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956.

The concept of closure algebra [McKinsey and Tarski, Ann. of Math. (2) 45 (1944), 141-191; 47 (1946), 122-162; MR 5, 211; 7, 359] is extended to general distributive lattices, and the class of lattices of all closed subspaces of a general  $T_0$ -space characterized abstractly as the class of "Kuratowski lattices". Various separation axioms are then interpreted lattice-theoretically, as well as compactness (Wallman). "Filters" (H. Cartan) alias "dual ideals" (M. H. Stone) alias "hereditary additive families" (C. Kuratowski) are also discussed.

G. Birkhoff.

Benado, Mihail. Bemerkungen zu einer Arbeit von Øystein Ore. Rev. Math. Pures Appl. 1 (1956), no. 2, 5-12.

As shown by the reviewer [Bull. Amer. Math. Soc. 49 (1943), 558-566; MR 5, 88] an analogue of the Jordan theorem about the equal length of maximal chains will hold in a partially ordered set under certain conditions, provided the theorem holds for the so-called simple cycles. The present writer replaces the simple cycles by a quadrilateral condition on four elements to improve the form of these theorems.

O. Ore.

Sasaki, Usa. Lattices of projections in  $AW^*$ -algebras. J. Sci. Hiroshima Univ. Ser. A. 19 (1955), 1-30.

It has been shown by Kaplansky [Ann. of Math. (2) 53 (1951), 235-249; MR 13, 48] that the lattice of projections of a finite  $AW^*$ -algebra forms a continuous geometry. The present paper is devoted to an extension to arbitrary  $AW^*$ -algebras. It is shown that these algebras have a dimension function and have lattices of finite projections forming what is known as a general continuous geometry [F. Maeda, J. Sci. Hiroshima Univ. Ser. A. 14 (1950), 85-92; MR 13, 313].

E. L. Griffin.

Andreoli, Giulio. Automorfismi in un algebra di Boole determinati da funzioni algebriche e trascendenti invertibili e gruppo dell'ipercubo. Ricerca, Napoli 6 (1955), no. 2, 3-9; no. 3, 3-7.

As before [Giorn. Mat. Battaglini (5) 3(83) (1955), 13-40; MR 17, 398], the author identifies the elements of the Boolean algebra  $2^n$  with the vertices of an  $n$ -cube  $C_n$ . The reflections of  $C_n$  give the translation group of the associated Boolean ring; the holomorph of this gives the group of automorphisms of  $2^n$  with respect to Glivenko's distance (and to Kiss' ternary median operation); those leaving 0 fixed are the automorphisms of  $2^n$  (and the translation group).

G. Birkhoff (Cambridge, Mass.).

See also: Kurepa, p. 270; Rogers, p. 271; Henkin, p. 272; Jakubík, p. 275; Zappa, p. 280; Fadini, p. 284; Choquet, p. 288; Bonsall, p. 320; Vitalbi, p. 326; Nedelcu, p. 368; Moisil et Ioanin, p. 368.

### Rings, Fields, Algebras

Jakubík, Ján. On the existence algebras. Časopis Pěst. Mat. 81 (1956), 43-54. (Czech. Russian and English summaries)

Let  $\mathfrak{A} = \mathfrak{A}(n_1, \dots, n_r)$  be the set of all "algebras" having  $r$  single-valued, universally defined  $n_i$ -ary operations  $f_1, \dots, f_r$ , in the sense of the reviewer [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; MR 10, 673] and Malcev [Mat. Sb. N.S. 35(77) (1954), 3-20; MR 16, 440]. If  $n_1 \geq 2$ , then  $\mathfrak{A}$  contains a "strongly non-permutable" algebra  $A$ , such that for each  $a \in A$  there exist congruence relations  $R, S$  on  $A$  and an infinity of pairs  $b, c \in A$  for which  $aRbSc$ , but  $aSxRb$  for no  $x \in A$ . This sharpens a result of Trevisan [Rend. Sem. Mat. Univ. Padova 22 (1953), 11-12; MR 15, 681]. Moreover, there exist congruence relations  $R \neq S$  on some  $A \in \mathfrak{A}$  such that  $aR=aS$  for some  $a \in A$ .

G. Birkhoff (Cambridge, Mass.).

Hazanov, M. B. On completeness of the field of real numbers. Kabardinskii Gos. Ped. Inst. Uč. Zap. 8 (1955), 19-20. (Russian)

**Cohn, Richard M.** An invariant of difference field extensions. Proc. Amer. Math. Soc. 7 (1956), 656-661.

This paper introduces a new numerical invariant, the limit degree, for certain extensions of difference fields. The principal theorem is: Let  $\mathfrak{F}$  be an extension of the difference field  $\mathfrak{F}$  and let  $\mathfrak{G}$  be a difference field such that  $\mathfrak{F} \subseteq \mathfrak{G} \subseteq \mathfrak{H}$ . Then the limit degree of  $(\mathfrak{F}/\mathfrak{G})$  is the product of the limit degree of  $(\mathfrak{G}/\mathfrak{F})$  by the limit degree of  $(\mathfrak{H}/\mathfrak{G})$ .

H. Levi (New York, N.Y.).

**Skopin, A. I.**  $p$ -extensions of a local field containing roots of unity of degree  $p^m$ . Izv. Akad. Nauk SSSR. Ser. Mat. 19 (1955), 445-470. (Russian)

Soient  $k$  un corps de nombres  $p$ -adiques contenant les racines  $p$ -ièmes primitives de l'unité (où  $p \nmid p$ ),  $n_0$  son degré absolu,  $m$  le plus grand entier tel que  $k$  contient les racines  $p^m$ -ièmes primitives de l'unité. Soit  $k_1$  l'extension de  $k$  définie par la récurrence suivante:  $k_0 = k$  et  $k_{i+1}$  est le composé de toutes les extensions cycliques de degré  $p$  de  $k_i$ . Soit  $k_\infty = \bigcup_i k_i$  la  $p$ -extension maximale de  $k$ . On notera  $G_{0/q}$  le groupe de Galois d'une extension  $Q/q$ . Si  $G$  est un groupe, notons  $G^p$  le groupe engendré par les puissances  $p$ -ièmes des éléments de  $G$ ,  $(a, b)$  le commutateur des  $a, b \in G$ ,  $[A, B]$  le groupe engendré par les  $(a, b)$ ,  $a \in A, b \in B$ . Soit  $G' = G^p[G, G]$  et, en général,  $G^{(i+1)} = (G^{(i)})'$ . Soit  $F_v$  le groupe fondamental d'une surface orientable de genre  $v$ , c'est-à-dire le quotient du groupe libre  $S_{2v}$  à  $2v$  générateurs  $a_1, a_2, \dots, a_v, b_1, b_2, \dots, b_v$  par le groupe engendré par le seul élément

$$(a_1, b_1)(a_2, b_2) \cdots (a_v, b_v).$$

L'auteur démontre que  $G_{k_\infty/k} = G_{k_\infty/k}/G_{k_\infty/k}^m$  est isomorphe à  $F_{(n_0+2)/2}/F_{(n_0+2)/2}^{(m)}$  Kawada [J. Fac. Sci. Univ. Tokyo. Sect. I. 7 (1954), 1-18; MR 16, 6] a démontré que  $G_{k_\infty/k}$  est le completé, par rapport à sa  $p$ -topologie, de  $S_{n_0+2}/H$ , où  $H$  est le groupe engendré par un seul élément  $h = uv^p$  tel que  $u \in [S_{n_0+2}, S_{n_0+2}]$ . Ainsi, le résultat de l'auteur est moins général (car il ne concerne que le facteur  $G_{k_\infty/k}$  de  $G_{k_\infty/k}$ ), mais plus précis (car il montre que  $u = \prod_{i=1}^{(n_0+2)/2} (a_i, b_i) \pmod{S_{n_0+2}}$  si les  $a_i, b_i$  sont des générateurs convenables de  $S_{n_0+2}$ ) que celui de Kawada.

La démonstration de ce résultat se décompose en plusieurs parties. Supposant démontrée l'existence dans  $G_{k_\infty/k}$  d'une relation de la forme  $\prod_{i=1}^{(n_0+2)/2} (a_i, b_i) = 1$ , (qui est triviale si  $m=1$ ), les propriétés des sous-groupes d'indice fini de  $F_v$  (à savoir que tout sous-groupe d'indice fini  $j$  de  $F_v$  est isomorphe à  $F_{1+j(v-1)}$ ) permettent de démontrer, par une méthode analogue à celle de Safarevič [Mat. Sb. N.S. 20(62) (1947), 351-363; MR 8, 560; 12, 1001] que  $(k_m:k) = (F_{(n_0+2)/2}/F_{(n_0+2)/2}^{(m)})$ , d'où résulte le théorème. D'autre part si  $m > 1$ , un lemme de la théorie des  $p$ -groupes permet de remplacer, pour ce problème d'existence, l'extension  $k_m/k$  par la plus grande surextension centrale  $\bar{k}_m/k$  de  $k_1/k$ , qui y est contenue, et certaines considérations simples de la théorie des corps permettent de réduire encore le problème en remplaçant  $\bar{k}_m/k$  par le composé  $k_2/k$  de toutes les surextensions centrales de  $k_1/k$  de degré  $p$ . La partie essentielle de la démonstration est précisément celle de l'identité  $\prod_i (a_i, b_i) = 1$  dans  $G_{k_1/k}$  pour un choix convenable des générateurs  $a_i$  et  $b_i$  de ce groupe.

Soit  $G = S_0/L$  une extension centrale d'un groupe  $\Gamma = S_0/H$  par un  $p$ -groupe élémentaire  $g = H/L$  (ce qui exige que  $L \supseteq H^p[H, S_0]$ ) et soit  $c$  la classe de 2-cohomologie de  $\Gamma$  opérant trivialement dans  $g$ , qui correspond à  $G$ .

Si  $\theta$  est un homomorphisme naturel de  $g$  et si  $n_0$  est son noyau,  $G/n_0$  est l'extension de  $\Gamma$  par  $g/n_0$ , dont la classe de cohomologie est le transformé  $c^\theta$  de  $c$  par  $\theta$ . Si  $\chi$  est un caractère de  $g$ , il transforme  $c$  en une classe de 2-cohomologie (triviale)  $c\chi$  de  $\Gamma$  dans le groupe  $E_p$  des racines  $p$ -ièmes de l'unité et  $\chi \rightarrow c\chi$  est un isomorphisme du groupe  $X_g$  des caractères de  $g$  sur un sous-groupe  $H_2^{(g)}$  du groupe  $H_2(\Gamma; E_p)$  de 2-cohomologie triviale de  $\Gamma$  dans  $E_p$ .  $X_g/n_0$  étant considéré comme un sous-groupe de  $X_g$ , on a, pour tout  $\chi \in X_g/n_0$ ,  $(c^\theta)\chi = c\chi$ . En particulier, si  $a_\chi$  est le noyau de  $\chi$  et si  $\theta_\chi$  est l'homomorphisme naturel de  $g$  sur  $g/a_\chi$ , on a  $c\chi = (c^\theta)\chi$ . Si  $n$  est un sous-groupe de  $g$ , on a donc  $c^{X_g/n} = H_2^{(g/n)} = \bigcup_{\tilde{g}} H_2^{(\tilde{g})}$ , où  $\tilde{g}$  parcourt tous les facteurs de degré  $p$  de  $g/n$ . Ainsi, si  $X_g$  est connu, un critère explicite pour que  $c\chi$ , où  $\chi \in X_g$ , appartienne à quelque  $H_2^{(\tilde{g})}$ , où  $\tilde{g}$  parcourt tous les facteurs d'ordre  $p$  de  $g/n$ , permet de déterminer  $X_g/n$  et, par dualité,  $n$ . Si  $L = H^p[H, S_{n_0+2}]$ , on a  $H_2^{(g)} = H_2(\Gamma; E_p)$ .

Soit  $\bar{G} = G_{k_1/k}$ . Alors,  $G_{k_1/k} = G_{k_1/k}/G_{k_1/k_1}$  est un  $p$ -groupe élémentaire à  $n_0+2$  générateurs et  $G$  est une extension centrale de  $\Gamma = G_{k_1/k}$  par  $g = G_{k_1/k_1} = G'_{k_1/k}$ . Comme  $g$  est le groupe de Frattini de  $G$ ,  $G$  a aussi  $n_0+2$  générateurs, et on peut poser  $G = S_{n_0+2}/N$  et  $\Gamma = S_{n_0+2}/H$ , où  $H = S_{n_0+2}'$  et  $N \supseteq H^p[H, S_{n_0+2}]$ . Quand  $\bar{k}/k_1$  parcourt toutes les extensions de degré  $p$  telles que  $\bar{k}/k$  soit une surextension centrale de  $k_1/k$ ,  $\bar{g} = G_{\bar{k}/k}$  parcourt tous les facteurs de degré  $p$  de  $g$ . Les propriétés des extensions kummériennes montrent qu'un  $t \in H_2(\Gamma; E_p)$  appartient à quelque  $H_2^{(\tilde{g})}$  si, et seulement si  $t \sim 1$  dans  $k_1$ , c'est-à-dire si l'invariant (écrit multiplicativement) du produit croisé  $(k_1/k, \Gamma = G_{k_1/k}; t)$  est  $= 1$ . Posons, d'autre part,  $L = H^p[H, S_{n_0+2}]$ ,  $G = S_{n_0+2}/L$ ,  $g = H/L$ ,  $n = N/L$  et soit  $c$  la classe de 2-cohomologie de l'extension  $G$  de  $\Gamma$  par  $g$ . Alors,  $n = \bigcap_i a_{\chi_i}$ , où  $\chi_i$  parcourt les caractères de  $g$  tels que l'invariant  $(k_1, c\chi_i) = (k_1/k, \Gamma; c\chi_i)$  est  $= 1$ .  $\chi \rightarrow (k_1, c\chi)$  est un isomorphisme de  $X_g$  dans  $E_p$ .

Soient  $\sigma_1^*, \sigma_2^*, \dots, \sigma_{n_0+2}^*$  un système complet minimal de générateurs de  $G$  et soient  $\sigma_1, \sigma_2, \dots, \sigma_{n_0+2}$  les éléments de  $\Gamma = G/g = \bar{G}/\bar{g} = G_{k_1/k_1}$  qui leur correspondent. Si  $k^*$  est le groupe multiplicatif de  $k$  et si  $\varepsilon \neq 1$  est une racine  $p$ -ième fixée de l'unité, il existe un et un seul élément  $\alpha_i$  de  $k^*/k^{*p}$  tel que  $\delta_i = \alpha_i^{1/p}$  soit invariant par les  $\sigma_j \neq \sigma_i$  et tel que  $\delta_i \sigma_i^{-1} = \varepsilon$ . Les  $\sigma_i^{*p}$  et les  $(\sigma_i^*, \sigma_j^*)$  ( $i, j = 1, 2, \dots, n_0+2$ ) constituent une base de  $g$ . Si  $\chi_i \in X_g$  resp.  $\chi_{ij} \in X_g$  est  $= 1$  sur tout élément de cette base sauf  $\sigma_i^{*p}$  resp.  $(\sigma_i^*, \sigma_j^*)$ , où il est  $= \varepsilon$ , ces caractères constituent une base de  $X_g$ , et il suffit de calculer  $(k_1, c\chi)$  pour  $\chi = \chi_i$  ou  $\chi_{ij}$ . L'auteur montre que  $(k_1, c\chi_i) = (k_1, \delta_i)$ ,  $\sigma_i^{*p} = \varepsilon$  et que  $(k_1, c\chi_{ij}) = (k_1, \delta_i, \delta_j)$ ;  $\sigma_i^{*p} = \sigma_j^{*p} = 1$ ,  $(\sigma_i, \sigma_j) = \varepsilon$ . Or, tous ces invariants s'expriment à l'aide des  $(\alpha_i, \varepsilon)$  et des  $(\alpha_i, \alpha_j)$ , où  $(\alpha, \beta)$  est le symbole des restes normiques d'Hilbert dans  $k$ . Puisque  $m > 1$ , on a  $\varepsilon \in k^{*p}$ , d'où  $(\alpha_i, \varepsilon) = 1$ . Soit  $\chi = \prod_i \chi_i^{\varepsilon_i} \prod_{i,j} \chi_{ij}^{\varepsilon_{ij}}$ . Alors, si l'on fait un choix biorthogonal des  $\alpha_i$  [c'est-à-dire tel que  $(\alpha_i, \alpha_j) = 1$  sauf si  $j = n_0+2-i$  et que  $(\alpha_i, \alpha_{n_0+2-i}) = \varepsilon$ ], on constate que  $(k_1, c\chi) = 1$  si, et seulement si

$$\zeta = \sum_i \varepsilon_i \alpha_{n_0+2-i} = 0 \pmod{p}.$$

Or,  $c\chi$  est précisément la valeur de  $\chi$  sur

$$h = \prod_i (\sigma_i^*, \sigma_{n_0+2-i}^*),$$

qui appartient, ainsi, au dual  $n$  de  $X_g$ . Par suite, si l'on pose  $a_i = \sigma_i^*$ ,  $b_i = \sigma_{n_0+2-i}^*$  ( $i = 1, 2, \dots, (n_0+2)/2$ ),  $\prod_i (a_i, b_i) = 1$  est bien une relation dans  $G/n = \bar{G} = G_{k_1/k}$ .  
M. Krasner (Paris).

**Abhyankar, Shreeram.** Two notes on formal power series. Proc. Amer. Math. Soc. 7 (1956), 903-905.

This paper consists of two separate parts. In the first it is shown by a simple construction that when  $n > 1$ , the ring  $k[[x_1, x_2, \dots, x_n]]$  of formal power series in  $n$  indeterminates over an arbitrary field  $k$  contains an infinite number of analytically independent elements. The second part calls attention to the following factorizations (characteristic  $p \neq 0$ ):

$$Z^p - Z - X^{-1} = \prod_{i=0}^{p-1} (Z + i - \sum_{j=1}^{\infty} X^{-j}/p^j),$$

$$Z^p - X^{p-1}Z - 1 = \prod_{i=0}^{p-1} (Z + iX - \sum_{j=1}^{\infty} X^{1-j}/p^j).$$

H. T. Muhly (Iowa City, Iowa).

**Abhyankar, Shreeram.** On the compositum of algebraically closed subfields. Proc. Amer. Math. Soc. 7 (1956), 905-907.

An example is constructed with the help of formal power series to show that the compositum of all the algebraically closed subfields of a field  $K$  need not be algebraically closed. However, if  $k$  is algebraically closed and if  $K$  is any finitely generated extension of  $k$ , then the compositum in question is algebraically closed because it is equal to  $k$ .

H. T. Muhly (Iowa City, Iowa).

**Rivoire, Paul.** Fonctions rationnelles sur un corps fini. Ann. Inst. Fourier, Grenoble 6 (1955-1956), 121-124.

Let  $K$  be a field,  $\Delta$  a simple transcendental extension of  $K$ , and  $X$  a field generator of  $\Delta/K$ . The  $K$ -automorphisms of  $\Delta$  are determined by the mappings:

$$X \rightarrow (aX + b)(cX + d)^{-1},$$

where  $a, b, c, d$  lie in  $K$ , and  $ad - bc \neq 0$  [see van der Waerden, *Moderne Algebra*, v. 1, 2nd ed., Springer, Berlin, 1937, pp. 206-207]. If  $G$  denotes the group of all  $K$ -automorphisms of  $\Delta$ , and if  $S$  is any subgroup, then  $K_S$  denotes the subfield of elements of  $\Delta$  left fixed by each automorphism of  $S$ . When  $K$  has infinitely many elements, then  $K_G = K$ . Otherwise  $K$  has  $q = p^n$  elements, where  $p$  is the prime characteristic of  $K$ , and  $K_G$  properly contains  $K$ . In fact,  $\Delta$  has degree over  $K_G$  equal to  $q^3 - q$ , the order of  $G$ . In this case, the author shows that  $K_G$  is obtained from  $K$  by the field adjunction of the element  $(X^q - X)^{q+1}(X^q - X)^{-(q^2+1)}$ . The author also presents explicit constructions of the fixed fields of nine subgroups of  $G$ . For example, let  $H, H'$ , and  $H_1$  denote the subgroups (of orders  $q(q-1)$ ,  $q$ , and  $q-1$ ) determined by the automorphisms  $\{X \rightarrow aX + b\}$ ,  $\{X \rightarrow X + b\}$ , and  $\{X \rightarrow aX\}$ , respectively. Then,  $(X^q - X)^{q-1}$  is the field generator of  $K_H$  over  $K$ ,  $K_{H'} = K(X^q - X)$ , and  $K_{H_1} = K(X^{q-1})$ .

C. C. Faith (East Lansing, Mich.).

★ **Guérindon, J.** Sur les chaînes maximales d'idéaux dans les anneaux. Séminaire A. Châtelet et P. Dubreil de la Faculté des Sciences de Paris, 1953/1954. Algèbre et théorie des nombres. 2e tirage multigraphié, pp. 10-01 - 10-11. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956.

Exposé d'ensemble sur les chaînes d'idéaux (quelconques, premiers ou primaires) dans les anneaux commutatifs. On dit qu'un idéal  $J$  d'un anneau commutatif  $A$  couvre un idéal  $I$  de  $A$  si  $ICJ, I \neq J$  et s'il n'existe aucun idéal entre  $I$  et  $J$ ; pour qu'il existe un idéal  $J$  couvrant  $I$ , il faut et il suffit qu'il existe un idéal maximal  $M$  de  $A$  tel que  $I:M \neq I$ , c'est à dire, dans le cas noethérien, qu'un

des idéaux premiers associés de  $I$  soit maximal; étude du quotient  $J/I$  dans le cas où  $J$  couvre  $I$ : ce quotient est, soit un corps, soit un anneau de carré nul, selon que  $J^2$  est contenu ou n'est pas contenu dans  $I$ ; application aux idéaux minimaux d'un anneau artinien. On dit qu'un idéal  $Q$  de  $A$  est quasi maximal si tout idéal premier contenant  $A$  est maximal; étude de ces idéaux dans le cas noethérien; dans un anneau noethérien  $A$  l'intersection des idéaux quasi maximaux contenant un idéal  $I$  est  $I$  lui-même; d'où, en prenant les idéaux quasi maximaux de  $A$  comme voisinages fondamentaux de 0, une topologie pour laquelle tout idéal est fermé; cette topologie est la topologie classique lorsque  $A$  est semi-local; l'auteur affirme, de façon erronée, que tout idéal  $I'$  du complété  $A'$  de  $A$  est de la forme  $A'I$  où  $I$  est un idéal de  $A$  (contre exemple: une variété algébrique non algébrique); son affirmation que le complété  $A'$  de  $A$  (supposé noethérien) est aussi noethérien est aussi fausse (lorsque  $A$  est l'anneau  $Z$  des entiers,  $A'$  est le produit de tous les anneaux  $p$ -adiques).}

P. Samuel (Clermont-Ferrand).

**Guérindon, Jean.** Théorie multiplicative des idéaux. C. R. Acad. Sci. Paris 243 (1956), 936-939.

Using an earlier note [same C. R. 242 (1956), 2693-2695; MR 18, 8] which introduced various topologies on the set of ideals of a commutative ring, the author now generalizes the classical multiplicative ideal theory to an arbitrary commutative ring.

A. Rosenberg.

**Andrunakievich, V. A.** Rings with annihilator condition. Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 547-568. (Russian)

The author subjects to close scrutiny the following class of rings (called  $A$ -rings): rings in which every minimal two-sided ideal  $I$  with  $I^2 \neq 0$  is a direct summand. The main point is to show how auxiliary hypotheses induce certain direct sum decompositions. These auxiliary hypotheses are mainly chain conditions on one-sided, two-sided or principal ideals.

I. Kaplansky.

**Rees, D.** A theorem of homological algebra. Proc. Cambridge Philos. Soc. 52 (1956), 605-610.

Soient  $A$  un anneau (avec unité),  $g_i (1 \leq i \leq k)$  des éléments du centre de  $A$ ,  $g_i$  l'idéal (bilatère) engendré par  $g_1, \dots, g_i$  (on convient que  $g_0 = (0)$ ); soit  $B$  l'anneau  $A/g_k$ . Soit  $N$  un  $A$ -module à gauche, satisfaisant à la condition

$$\Gamma(N, g_i): \text{ pour tout } i, (x \in N \text{ et } g_i x \in g_{i-1}N) \Rightarrow (x \in g_{i-1}N).$$

Alors, si  $M$  est un  $B$ -module à gauche et si  $A$  satisfait à  $\Gamma(A, g_i)$ , on a

$$(1) \quad \text{Ext}_A^n(M, N) \approx \text{Ext}_B^{n-k}(M, N/g_k N),$$

ce qui signifie notamment que  $\text{Ext}_A^n(M, N) = 0$  pour  $n < k$ ; l'isomorphisme est canoniquement défini quand la suite des  $g_i$  est choisie.

L'auteur énonce ce résultat en supposant  $A$  commutatif, et le prouve par récurrence sur  $k$ . (On pourrait le déduire de la suite spectrale

$$\text{Ext}_B^p(M, \text{Ext}_A^q(B, N)) \Rightarrow \text{Ext}_A^{p+q}(M, N)$$

(Cartan-Eilenberg, *Homological Algebra*, p. 349, formule (2)<sub>4</sub>), et du fait que, vu les conditions  $\Gamma(A, g_i)$  et  $\Gamma(N, g_i)$ , on a  $\text{Ext}_A^q(B, N) = 0$  pour  $q \neq k$ ,  $\text{Ext}_A^k(B, N) = N/g_k N$ .) L'auteur donne aussi une version „graduée” de son théorème. Il l'applique à la structure des idéaux d'un anneau noethérien  $A$ : un idéal  $g$  est dit „général de rang

$k''$  s'il possède un système de générateurs  $g_1, \dots, g_k$  satisfaisant à  $\Gamma(A, g_i)$ . Soit  $a$  un idéal,  $a \supset g$ ; alors  $(g:a)/g = \text{Hom}_A(A/a, A/g)$  est, d'après (1), isomorphe à

$$\text{Ext}_A^k(A/a, A),$$

indépendant de  $g$ , tandis que  $\text{Ext}_A^n(A/a, A) = 0$  pour  $n < k$ . On en déduit: si  $g'$  général de rang  $k' < k$  est contenu dans  $a$ ,  $a$  contient un  $g_1$  général de rang  $k$  tel que  $g_1 \supset g'$ ; le plus grand entier  $k$  tel que  $a$  contienne un idéal général de rang  $k$  est le plus petit  $k$  tel que

$$\text{Ext}_A^k(A/a, A) \neq 0.$$

H. Cartan (Paris).

Hochschild, G. Relative homological algebra. Trans. Amer. Math. Soc. 82 (1956), 246-269.

L'auteur développe une théorie „relative” des foncteurs Tor et Ext de Cartan-Eilenberg [Homological algebra, Princeton, 1956, cité ci-dessous HA; MR 17, 1040]; il en donne des applications à la cohomologie relative des algèbres, des groupes, des algèbres de Lie.

Soit  $\varphi: S \rightarrow R$  un homomorphisme d'anneaux à élément unité. Un  $R$ -module  $A$  est  $\varphi$ -projectif (HA, p. 30) si et seulement si le foncteur  $\text{Hom}_R(A, X)$  du  $R$ -module  $X$  transforme toute suite  $(R, S)$ -exacte de  $R$ -modules en une suite exacte de groupes abéliens (on appelle suite  $(R, S)$ -exacte une suite de  $R$ -modules et de  $R$ -homomorphismes qui est exacte et admet une  $S$ -homotopie). Caractérisation analogue des  $R$ -modules  $\varphi$ -injectifs. Pour tout  $R$ -module  $A$ , on appelle  $(R, S)$ -résolution projective de  $A$  une suite  $(R, S)$ -exacte

$$\cdots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 \rightarrow A$$

telle que les  $X_n$  soient  $\varphi$ -projectifs; on définit de même les  $(R, S)$ -résolutions injectives. Ces résolutions servent à définir les foncteurs dérivés  $\text{Tor}_n^{(R,S)}(A, B)$  de  $A \otimes_R B$ , et  $\text{Ext}_{(R,S)}^n(A, B)$  de  $\text{Hom}_R(A, B)$ . On a des homomorphismes naturels  $\text{Tor}_n^R(A, B) \rightarrow \text{Tor}_n^{(R,S)}(A, B)$ ,  $\text{Ext}_{(R,S)}^n(A, B) \rightarrow \text{Ext}_R^n(A, B)$ , qui sont des isomorphismes quand l'anneau  $S$  est semi-simple.

Application aux algèbres: soient  $P$  et  $Q$  deux  $K$ -algèbres,  $\psi: Q \rightarrow P$  un homomorphisme de  $K$ -algèbres. Posons  $R = P \otimes_K P$ ,  $S = Q \otimes_K P$ ,  $T = Q \otimes_K Q'$  où  $P'$  et  $Q'$  désignent les algèbres „opposées” à  $P$  et  $Q$ . Si  $M$  est un  $P$ -bimodule, considéré comme  $(P \otimes P)$ -module à gauche, on sait (HA, Chap. IX) que (au moins lorsque  $K$  est un corps) les groupes de cohomologie de Hochschild  $H^n(P; M)$  ne sont autres que les  $\text{Ext}_R^n(P, M)$ , où  $P$  est considéré comme  $P$ -bimodule. L'auteur définit ici (même si  $K$  n'est pas un corps) les groupes „relatifs”

$$H^n(P, Q; M) = \text{Ext}_{(R,S)}^n(P, M)$$

[Remarque du rapporteur: ces groupes sont aussi égaux à  $\text{Ext}_{(R,T)}^n(P, M)$ ]. On peut les calculer au moyen d'un complexe analogue au „complexe standard”: on prend  $X_n(P, Q) = P \otimes_Q P \otimes \cdots \otimes_Q P$  ( $n+2$  facteurs) avec sa structure naturelle de  $P$ -bimodule, et avec les formules habituelles donnant l'opérateur différentiel  $X_n(P, Q) \rightarrow X_{n-1}(P, Q)$ . La différence avec le complexe standard réside dans le fait que les produits tensoriels sont pris sur  $Q$  au lieu de  $K$ , et on a en particulier  $H^n(P; M) = H^n(P, K; M)$  sans hypothèse sur  $K$ . [Le rapporteur observe qu'il y a aussi un complexe „normalisé”]

$$\bar{X}_n(P, Q) = P \otimes_Q \bar{P} \otimes \cdots \otimes_Q \bar{P} \otimes_Q P,$$

avec  $n$  facteurs  $\bar{P}$ , où  $\bar{P}$  désigne le  $Q$ -bimodule, quotient du  $T$ -module  $P$  par l'image de  $Q$ .

Dimension relative: soit  $c(P, Q)$  la borne sup. des  $n$  tels qu'il existe un  $M$  avec  $H^n(P, Q; M) \neq 0$ ; soit  $d(P, Q)$  la borne sup. des  $n$  tels qu'il existe des  $P$ -modules à gauche  $U, V$  avec  $\text{Ext}_{(P,Q)}^n(U, V) \neq 0$ . On démontre  $d(P, Q) \leq c(P, Q)$ ,  $c(P, Q) = d(P \otimes P', P \otimes Q')$ ; en fait,  $c(P, Q)$  est aussi égal à  $d(P \otimes P', Q \otimes Q')$ . Or  $d(P, Q) = 0$  signifie que toute  $P$ -suite exacte qui admet une  $Q$ -homotopie admet une  $P$ -homotopie; il en résulte notamment que si  $c(P, Q) = 0$ , tout  $P$ -bimodule qui est semi-simple comme  $Q$ -bimodule l'est comme  $P$ -bimodule.

Cohomologie relative des groupes: soient  $G$  un groupe,  $K$  un sousgroupe; prenons  $R = Z(G)$ ,  $S = Z(K)$ ; pour tout  $R$ -module  $M$ , on définit les groupes de cohomologie relatifs  $H^n(G, K; M) = \text{Ext}_{(R,S)}^n(Z, M)$ , où  $Z$  est considéré comme  $R$ -module au moyen de l'augmentation unitaire  $Z(G) \rightarrow Z$ . Ces groupes relatifs coïncident avec ceux définis récemment par Adamson [Proc. Glasgow Math. Assoc. 2 (1954), 66-76; MR 16, 442]; ils peuvent être calculés avec un complexe analogue au classique „complexe homogène”; (il y a aussi un „complexe non homogène” que l'auteur ne mentionne pas). Si  $K$  est sous-groupe invariant de  $G$ ,  $H^n(G, K; M) = H^n(G/K; M^K)$ ,  $M^K$  désignant le sous-module des éléments de  $M$  invariants par  $K$ . Le groupe  $H^2(G, K; M)$  s'interprète au moyen de certaines classes d'extensions de groupes. Lorsque  $K$  est d'indice fini dans  $G$ , il existe une „résolution complète” donnant naissance à des groupes de cohomologie relatifs  $\tilde{H}^n(G, K; M)$  pour les  $n$  entiers  $\geq 0$  ou  $< 0$ ; on a une suite exacte analogue à celle de Tate; on retrouve ainsi les groupes d'Adamson. Il y a aussi des groupes d'homologie  $\tilde{H}_n(G, K; M)$  pour  $n \geq 0$  ou  $< 0$ , mais, contrairement au cas absolu, on n'a pas, en général,  $\tilde{H}^{-n} = \tilde{H}_{n-1}$ .

Cohomologie relative des algèbres de Lie: soient  $L$  une algèbre de Lie sur un corps  $F$ ,  $K$  une sous-algèbre de Lie. On prend pour  $R$  et  $S$  les algèbres enveloppantes universelles de  $L$  et  $K$ ; si  $M$  est un  $L$ -module, on définit les groupes de cohomologie relatifs

$$H^n(L, K; M) = \text{Ext}_{(R,S)}^n(F, M),$$

où  $F$  est considéré comme  $L$ -module trivial. Le complexe standard relatif (construit avec  $R$  et  $S$ ) peut servir à calculer ces groupes, mais on désire un complexe plus petit, tel que le calcul des cochaines et du cobord fasse seulement intervenir les opérations de l'algèbre de Lie. Soit  $E(L/K)$  l'algèbre extérieure du  $F$ -module  $L/K$ ; on y fait opérer  $S$  à gauche, en posant

$$y \cdot (x_1 \wedge \cdots \wedge x_n) = \sum_{1 \leq i \leq n} x_1 \wedge \cdots \wedge [y, x_i] \wedge \cdots \wedge x_n,$$

$$y \in K, x_i \in L/K.$$

Alors  $R \otimes_S E(L/K)$  muni de l'opérateur différentiel donné par la formule classique, est un complexe  $(R, S)$ -projectif et acyclique; il se plonge naturellement dans le complexe standard relatif normalisé. Si l'on savait que  $R \otimes_S E(L/K)$  admet une  $S$ -homotopie, on pourrait conclure que les groupes de cohomologie qu'on calcule avec ce complexe (et qui sont ceux de Chevalley-Eilenberg) sont bien les  $H^n(L, K; M)$ ; c'est à là un problème ouvert. L'auteur le résout par l'affirmative lorsque  $F$  est de caractéristique 0 et  $K$  réductive dans  $L$  (alors  $R \otimes_S E(L/K)$  est  $S$ -semi-simple), et lorsque  $K$  est un idéal de  $L$ , auquel cas  $H^n(L, K; M) = \tilde{H}^n(L/K; M^K)$ ,  $M^K$  désignant le sous-module des éléments de  $M$  annulés par  $K$ . H. Cartan.

See also: Jaskowski, p. 271; Carlitz, p. 274; Boccioni, p. 283; Eichler, p. 299; Monna, p. 320; Bonsall, p. 320; Donoghue, p. 322; Vitalbi, p. 326.

## Groups, Generalized Groups

**Benado, Mihail.** Sur la théorie générale des produits réguliers. C. R. Acad. Sci. Paris 243 (1956), 1092-1093.

Let  $G$  be a group with  $\Sigma$  as operator domain, and  $G'$  be a  $\Sigma$ -subgroup. Denote by  $A \circ B$  the commutator of subgroups  $A$  and  $B$ , and let  $GO^*G'$  denote  $GO(GO^{*-1}G')$ .  $G'$  is called a normal divisor of degree  $h$  of  $G$  if  $GO^*G' \leq G'$ ;  $G$  is the direct product of degree  $h$  of its  $\Sigma$ -subgroups  $G_i$  if each  $G_i$  is a normal divisor of degree  $h$  of  $G$  and  $G$  is the regular product of the  $G_i$  [O. N. Golovin, Dokl. Akad. Nauk SSSR (N.S.) 58 (1947), 1257-1260; Mat. Sb. N.S. 27(69) (1950), 427-454; MR 9, 493; 12, 672]. For  $h=1$  these reduce to the ordinary definitions of normal divisor and direct product. Several properties are stated without proof. For example, every  $h$ -nilpotent product is a direct product of degree  $h+1$ ; a regular product is a direct product of degree  $h$  if and only if the  $h$ th commutator of its factors is the identity; a group (other than the identity) cannot be both a free product and a direct product of degree  $h$ .  
P. M. Whitman (Silver Spring, Md.).

**Whittaker, J. V.** On the postulates defining a group. Amer. Math. Monthly 62 (1955), 636-640.

The following three axioms characterize an (additively written) group in terms of the operation  $x-y$ : closure, existence of 0 such that  $x-y=0$  if and only if  $x=y$ ;  $(x-z)-(y-z)=x-y$ . Replacing the last by

$$(x-z)-(x-y)=y-z$$

gives commutativity. If  $G$  is a set closed under  $x-y$ , and  $E$  is an equivalence relation on  $G$  satisfying:  $(v-y)E(x-y)$  if and only if there exists  $z$  such that  $vE(x-z)$  and  $wE(y-z)$ , then  $R$  is a (substitutive) congruence relation and  $G$  modulo  $R$  is a group. [For related considerations and bibliography, see Boggs and Rainich, Bull. Amer. Math. Soc. 43(1937), 81-84.]  
R. C. Lyndon.

**Kuo, Ke-chan.** The imbedding problem for systems with an incomplete, commutative addition. J. Chinese Math. Soc. (N.S.) 1 (1951), 68-87. (Chinese summary)

The author continues, with special attention to the commutative case, the investigations of Baer [Amer. J. Math. 71 (1949), 706-742; MR 11, 78] on the embeddability of unions of groups in a group. A commutative "add"  $A$  is a set with a sometimes defined addition such that  $x+y=y+x$  whenever either is defined.  $A$  is the amalgam of its subgroups if  $x+y$  is defined only for  $x$  and  $y$  in a common subgroup. Amalgams with 0 and inverses are considered, as well as those where the existence of  $x+y$ ,  $y+z$ ,  $z+x$  implies that  $x$ ,  $y$ ,  $z$  lie in a common subgroup. Embeddability is studied by means of the semigroup  $D(A)$ , whose elements are formal products  $u=u_1 \cdots u_n$  of elements  $u_i$  from  $A$ , identified under the equivalence generated by direct reduction: replacing  $u_i u_{i+1}$  by  $u_i + u_{i+1}$  when the latter is defined.  $Dc(A)$  is  $D(A)$  made abelian. Conditions that the natural homomorphisms of  $A$  into  $D(A)$  and  $Dc(A)$  be one to one or isomorphisms are given in terms of relations between equivalence and direct reduction. Further such conditions, e.g. that two equivalent irreducible products have the same length, are examined, together with their implications for  $A$ . In particular, it is shown that certain of these conditions on  $D(A)$ , for which Baer obtained criteria, carry over to  $Dc(A)$ .  
R. C. Lyndon (Berkeley, Calif.).

★ **Aleksandrow, P. S.** Wstęp do teorii grup. [Introduction to the theory of groups.] Państwowe Wydawnictwo Naukowe, Warszawa, 1956. 153 pp. zł. 4.70. A translation by Robert Bartoszyński of the author's Vvedenie v teoriyu grupp [Učpedgiz, Moscow, 1951]. The German translation was reviewed in MR 16, 791.

**Muhammedžan, H. H.** On groups possessing increasing invariant sequence. Mat. Sb. N.S. 39(81) (1956), 201-218. (Russian)

The first chapter of this paper contains among others proofs of the following results: The center of a finite  $p$ -extension of a locally finite  $p$ -group  $P$  satisfies m.c.s. (minimal condition for subgroups) if and only if the center of  $P$  satisfies m.c.s. If an abelian normal subgroup  $A$  of a periodic  $ZA$ -group  $G$  satisfies m.c. for normal subgroups of  $G$ , then  $A$  satisfies m.c.s.; if moreover  $A$  is maximal then  $G$  satisfies m.c.s. [This generalizes a result of Černikov, Dokl. Akad. Nauk SSSR (N.S.) 72 (1950), 243-246; MR 12, 77.] If those factors with natural number indices of the a.c.s. (ascending central series) of a periodic  $ZA$ -group  $G$  satisfy m.c.s., then  $G$  satisfies m.c.s., and the length of the a.c.s. is a non-limit ordinal number  $< \omega_2$ . [This sharpens a result of the author, Mat. Sb. N.S. 28(70) (1951), 185-196; MR 12, 587.]

In the second chapter the author studies groups  $G$  possessing  $MA$ -series, i.e., well-ordered invariant series of subgroups  $1 \subset A_1 \subset \cdots \subset A_\alpha \subset \cdots \subset A_\gamma = G$  in which every factor  $A_{\alpha+1}/A_\alpha$  is a maximal abelian normal subgroup of  $G/A_\alpha$ . Main results: A group is a solvable group with m.c.s. if and only if it possesses an  $MA$ -series whose factors with natural number indices satisfy m.c.s. If the first two factors of any  $MA$ -series of a periodic group  $G$  satisfy m.c.s., then  $G$  is a solvable group with m.c.s. (An example is given showing that the conclusion of this theorem does not hold if only the first factor of an  $MA$ -series satisfies m.c.s.). If any one of the maximal abelian normal subgroup of a periodic group  $G$  satisfies m.c.s. and if  $G$  possesses an  $MA$ -series of length 2, then  $G$  satisfies m.c.s.  
R. Ree (Vancouver, B.C.).

**Fridman, M. A.** On semi-commutative multiplications. Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 710-712. (Russian)

The author considers a generalization of free and direct products for groups that may be described in the following manner: Let  $\Omega$  be a class of groups that is closed with respect to isomorphisms. Let  $T_1$  and  $T_2$  be operations defined on  $\Omega$  and satisfying: (i)  $T_1(G)$  and  $T_2(G)$  are normal subgroups of  $G$  for all  $G \in \Omega$ ; (ii) for isomorphic groups  $A$  and  $B$  in  $\Omega$ , there exists an isomorphism  $\varphi$  of  $A$  on  $B$  such that  $\varphi(T_1(A)) = T_1(B)$  ( $i=1, 2$ ). A group  $G$  is called a semi-commutative  $T$ -product of subgroups  $A_\alpha$ , where  $\alpha$  runs through a set  $\Sigma$ , if (i)  $A_\alpha \in \Omega$  ( $\alpha \in \Sigma$ ); (ii)  $G = \langle A_\alpha \rangle_{\alpha \in \Sigma}$ ; (iii) the set of relations  $\bigcup \Psi_{\alpha \in \Sigma} (A_\alpha) \cup \Psi$ , where  $\Psi(A_\alpha)$  is the set of all relations between elements of  $A_\alpha$  and  $\Psi$  is the set of relations of the form  $a^{-1}b^{-1}ab=1$  for  $a \in T_1(A_\beta)$ ,  $b \in T_1(A_\gamma)$  ( $\beta, \gamma \in \Sigma$ ,  $\beta \neq \gamma$ ;  $i, j=1, 2$ ,  $i \neq j$ ), represent the set of relations defined on  $G$  relative to a system of generators consisting of the elements of the groups  $A_\alpha$  ( $\alpha \in \Sigma$ ).

Several theorems are stated concerning the associativity of  $T$ -products and the composition of the center of  $G$  with respect to the  $T$ -product. Numerous illustrations of  $T$ -products are indicated.  
L. J. Paige.

**Masuda, Katsuhiko.** Generalization of the concept of cohomology of finite groups. *Proc. Japan Acad.* 31 (1955), 504-507.

Chevalley's definition of cohomology groups [Class field theory, Nagoya Univ., 1954; MR 16, 678] is extended to the definition of local cohomology groups at a rational prime  $l \leq \infty$  and a representation  $D$ , of degree  $d$ , of the finite group  $G$  on the integers. Rather than the lengthy definition, we present an interpretation by Serre, quoted in an addendum. Let  $A$  be a right  $G$  module,  $\bar{A}$  the inverse limit of the  $A/l^i A$ ,  $\bar{Z}$  the  $l$ -adic integers,  $\bar{G}$  the group ring of  $G$  over  $\bar{Z}$ ; let subscript  $d$  indicate the corresponding matrix rings of degree  $d$ . Then  $\bar{A}_d$  is a left  $\bar{Z}_d$  and right  $\bar{G}_d$  module. Let  $J = \bar{G}/\sum Gg$ ,  $I$  be the kernel of natural  $\bar{G}_d \rightarrow \bar{Z}_d$ , and  $K$  be the tensor product over  $\bar{Z}$  of  $p$  factors  $J$ ,  $q$  factors  $I$ , and one factor  $\bar{A}_d$ . Let  $G$  operate on  $\bar{Z}_d$  through  $M^g = D(g^{-1})M$ , hence on  $A' = K \otimes \bar{Z}_d$ , tensored over  $\bar{Z}_d$ . Then the author's groups are the  $pqH(A', G)$ .

R. C. Lyndon (Berkeley, Calif.).

**Weir, A. J.** The Reidemeister-Schreier and Kuroš subgroup theorems. *Mathematika* 3 (1956), 47-55.

Let a group  $G$  be generated by element  $\{x\}$  and relations  $r=1$ . Then  $G=X/R$ , where  $X$  is the free group generated by elements  $\{x\}$  and  $R$  is the "consequence" of the  $r$ , the least normal subgroup of  $X$  containing the  $r$ . We say that  $G$  has the presentation  $\{X; r=1\}$ . If  $H$  is a subgroup of  $G$ , let the left cosets of  $H$  be designated as  $H_b$ , where  $b$  runs over a set of indices and  $H_1=H$ . Also let  $\bar{H}_b$  be a representative for  $H_b$ . Then  $H$  is generated by the elements  $u_{bx} = \bar{H}_b \cdot x \cdot \bar{H}_b^{-1}$ . Consequently if we form a free group  $Y$  with generators  $y_{bx}$ , there will be a homomorphism  $Y \rightarrow H$  with  $y_{bx} \rightarrow u_{bx}$ . In the homomorphism  $X \rightarrow G=X/R$ , let  $W$  be the inverse image of  $H$ , and  $\bar{W}_b$  the coset of  $W$  which is the inverse image of  $H_b$ . It is shown that  $H$  has a presentation  $\{Y; u^W=1, r^W=1\}$ , where  $W=W_1$  and generally  $W_b$  are coset maps defined by the rules  $(uv)^{W_b} = u^{W_b}v^{W_b}$ ,  $1^{W_b}=1$ ,  $x^{W_b}=y_{bx}$ ,  $(v^{-1})^{W_b} = (v^{W_b})^{-1}$  for  $x$  a generator and  $u, v$  arbitrary elements of  $X$ . A transversal (complete set of coset representatives) is called a Schreier system if whenever  $a_1 \cdots a_t$  is the reduced form of a  $\bar{W}_b$ , then also  $a_1 \cdots a_{t-1}$  is also a  $\bar{W}_c$ . We may take a transversal which is a Schreier system, and then it is shown that the presentation of  $W$  takes the simple form  $\{Y; \text{certain } y_{bx}=1\}$ , proving the Schreier theorem that every subgroup  $W$  of a free group  $X$  is free.

The author also proves the Kurosch theorem that a subgroup  $H$  of a free product  $G = *G_\alpha$  is itself a free product of conjugates of subgroups of the  $G_\alpha$  and a free group  $F$ . The proof is similar to the proof of the Schreier theorem, but involves introducing further generators, choosing a number of transversals for  $H$  in  $G$ , one for each  $\alpha$ , and relating these transversals to each other.

Marshall Hall, Jr. (Columbus, Ohio).

**Kulikov, L. Ya.** On direct decompositions of a mixed abelian group. *Publ. Math. Debrecen* 4 (1956), 512-516. (Russian)

In a previous paper [Trudy Moskov. Mat. Obšč. 2 (1953), 85-167; MR 15, 9] the author gave a generalization of Ulm's theorem resting on three assumptions. It was known that the first and third of these were indispensable. It is now shown that the same is true for the second hypothesis (every Ulm factor is a direct sum of cyclic groups).

I. Kaplansky (Princeton, N.J.).

**Gluškov, V. M.** On the theory of nilpotent locally bicom- pact groups. *Izv. Akad. Nauk SSSR. Ser. Mat.* 20 (1956), 513-546. (Russian)

The author considers nilpotent bicom- pact groups possessing finite normal series with factors isomorphic to the additive group  $F_p^+$  of the field  $F_p$  of  $p$ -adic integers and establishes connections of such groups to nilpotent Lie algebras over the field  $F_p$  of  $p$ -adic numbers and to finitely generated abstract nilpotent groups.

A nilpotent Lie algebra  $L$  over  $F_p$  can be made into a topological group  $H$  with the binary operation defined by the Campbell-Hausdorff formula  $xy = x + y + \frac{1}{2}[x, y] + \cdots$ , and with the topology induced by the  $p$ -adic topology in  $F_p$ . The author characterizes nilpotent bicom- pact groups  $G$  which are locally isomorphic to  $H$  as those possessing  $p$ -adic series, i.e., a finite normal series whose factors are all isomorphic to  $F_p^+$  (in case some factors are finite cyclic  $p$ -groups the normal series is called a  $p$ -series), and shows that for such a group  $G$  the group  $H$  and the Lie algebra  $L$  are uniquely determined.  $L$  is called the Lie algebra of  $G$ .

A sequence  $A_1 \supset A_2 \supset \cdots$  of normal subgroups of an abstract group  $G$  is called a  $p$ -sequence if all  $G/A_i$  are  $p$ -groups. The completion of the topological group obtained from  $G/A$ , where  $A = \bigcap A_i$ , by taking  $\{A_i/A\}$  as a complete system of neighborhood of the unit is called a  $p$ -closure of  $G$ . If  $A_i$  is the subgroup generated by  $p^i$ -th powers of elements in  $G$ , then the  $p$ -closure is called canonical. Some results: Any  $p$ -closure of an abstract finitely generated nilpotent group is a nilpotent bicom- pact group with a  $p$ -series, and vice versa. The canonical  $p$ -closure of an arbitrary abstract finitely generated torsion-free nilpotent group is a nilpotent bicom- pact group with a  $p$ -adic series, and its Lie algebra has in a suitable basis rational structure constants. Any nilpotent bicom- pact group with a  $p$ -series (in particular, any finite group) is topologically isomorphic to a factor group of some nilpotent bicom- pact group with a  $p$ -adic series.

The last section is devoted to the study of solvable and nilpotent locally bicom- pact groups with m.c.s. (maximal condition for closed subgroups). Some results: Any solvable locally bicom- pact group with m.c.s. is an extension of a solvable bicom- pact group with m.c.s. by a discrete solvable group with m.c.s. A topological group is a solvable locally bicom- pact group with m.c.s. if and only if it has a finite normal series every factor of which is topologically isomorphic either to  $F_p^+$  for some prime  $p$  or to a discrete cyclic group, and where  $p$ -adic factors precede all infinite cyclic factors. A nilpotent locally bicom- pact group  $G$  with m.c. for (normal) subgroup has a finite central series

$$\{e\} = A_0 C \cdots C A_k = B_0 C \cdots C B_m = C_0 C \cdots C C_n = G,$$

where every  $A_i/A_{i-1}$  is finite cyclic, every  $B_j/B_{j-1}$  is topologically isomorphic to  $F_p^+$  for some prime  $p$ , and every  $C_\alpha/C_{\alpha-1}$  is an infinite discrete cyclic group. Finite direct products of nilpotent bicom- pact groups with  $p$ -series (with one or several primes  $p$ ) and only those are nilpotent bicom- pact groups with m.c.s. R. Ree.

**Zappa, Guido.** Sui gruppi finiti per cui il reticolo dei sottogruppi di composizione è modulare. *Boll. Un. Mat. Ital.* (3) 11 (1956), 315-318.

It is shown that in order that the lattice of composition subgroups be modular it is necessary and sufficient that for a normal chain these quotient groups which are of prime power order be quasi-abelian, that is, any two sub-

groups be permutable. For a so called supersolvable group to have this property it is necessary and sufficient that the Sylow groups be quasi-abelian. *O. Ore.*

**Szép, J.** Über endliche Gruppen, die nur einen echten Normalteiler besitzen. Acta Sci. Math. Szeged 17 (1956), 45-48.

The author proves: Let  $G$  be a finite group, and suppose that  $N \neq 1$ ,  $G$  is the only normal subgroup contained in  $G$ . In that case  $N$  is not a direct factor of  $G$  and one of the cases 1-3 holds: 1.  $G=HN$ ,  $H \cap N = (1)$ , where  $H$  is a simple group, and (a)  $N$  is simple, or (b)  $N$  is a nonsimple elementary abelian group and the simple subgroups of  $N$  are conjugate in  $G$ , or (c)  $N$  is the direct product of at least two simple groups of composite order which are conjugate in  $G$ . 2.  $N$  is the Frattini subgroup of  $G$ , is at the same time an elementary abelian group and the simple subgroups of  $N$  are conjugate in  $G$  and  $G/N$  is simple. 3.  $G=HN$ ,  $H \cap N = D \neq (1)$ , where  $N$  is the direct product of simple subgroups of composite order which are conjugate in  $G$  and  $D$  is the Frattini subgroup of  $H$  and  $H/D$  is simple. *I. N. Herstein (New Haven, Conn.).*

**Itô N.; und Szép, J.** Über nichtauflösbare endliche Gruppen. Acta Sci. Math. Szeged 17 (1956), 76-82.

If  $G$  is a finite group let  $r(G)$  be the number of non-isomorphic subgroups of  $G$  which are not normal in  $G$ , and let  $t(G)$  be the number of distinct primes which divide the order of  $G$ . The authors prove: Let  $G$  be a finite, unsolvable group with the property that  $r(G) \leq 3t(G) + 2$  which contains a non-trivial normal  $p$ -subgroup (where  $p$  is a prime); then  $G$  is isomorphic to one of the groups (a) the special linear homogeneous group  $SLH(2,5)$  or (b) the group  $A_5 \times Z_p$ , where  $A_5$  is the icosahedral group and  $Z_p$  is a cyclic group of order  $p$ , where  $p$  is not 2, 3 or 5.

This is related to a result of Itô who had shown that a finite nonsolvable group satisfying  $r(G) < 2t(G) + 2$  must be  $A_5$ . *I. N. Herstein (New Haven, Conn.).*

**Sysoev, A. E.** On decomposition of symmetric groups according to a double cyclic module and its application to the theory of textile interlacing. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 2(68), 209-214. (Russian)

A "chief weave" is one in which, in the basic pattern of  $r \times r$  threads, exactly one horizontal thread lies under each vertical thread, and exactly one vertical thread lies above each horizontal thread. Thus there is a many-one correspondence between permutations of degree  $r$  and the different chief waves. The main result is that if  $G$  is the subgroup of the symmetric group  $S^{(r)}$  generated by a permutation of order  $r$ , then the essentially different weaves correspond to the double cosets  $GaG$ ,  $a \in S^{(r)}$ . The author uses a formula of Frobenius [J. Reine Angew. Math. 101 (1887), 273-299] to calculate the number of different weaves, and other results.

*R. A. Rankin and D. H. McLain.*

**Steinberg, Robert.** Prime power representations of finite linear groups. Canad. J. Math. 8 (1956), 580-591.

L'auteur considère les groupes classiques finis (groupe projectif unimodulaire, groupe projectif symplectique, groupe projectif unitaire, groupe projectif orthogonal, sur un corps fini de caractéristique  $p$ ). Soit  $d$  l'ordre d'un  $p$ -groupe de Sylow d'un tel groupe  $G$ , et soit  $m$  l'indice dans  $G$  du normalisateur de ce groupe de Sylow; l'auteur montre que  $G$  admet une représentation irréductible de degré  $d$  sur tout corps de caractéristique 0 ou première à  $m$ .

Lorsque  $G = PSL_n(F_q)$ , l'idée directrice est la suivante: soit  $\mathcal{L}$  l'ensemble des combinaisons linéaires formelles à coefficients entiers des simplexes (ordonnés) dans l'espace projectif sur  $K$  de dimension  $n-1$  (i.e. les suites  $(S_1, \dots, S_n)$  de  $n$  points projectivement indépendants). Il est clair que  $G$  opère dans l'ensemble  $\mathcal{L}$ ; on définit dans  $\mathcal{L}$  une relation d'équivalence  $R$  compatible avec les opérations de  $G$ , de sorte que le module quotient  $\Sigma$  soit un module libre ayant une base de  $d$  éléments; on peut alors (par produit tensoriel) étendre les scalaires de  $\Sigma$  à un corps quelconque, ce qui fournit la représentation cherchée. La relation  $R$  est définie en considérant l'ensemble  $\mathcal{L}^*$  des drapeaux de l'espace projectif à  $n-1$  dimensions (suites  $(D_0, D_1, \dots, D_{n-1})$  où  $D_i$  est de dimension  $i$  et  $D_i \subset D_{i+1}$ ); pour un simplexe  $\Delta$  et un drapeau  $V$ , on pose  $(\Delta, V) = \varepsilon = \pm 1$  s'il existe une permutation  $\sigma$  des indices  $1, 2, \dots, n$  telle que  $D_i$  soit la variété linéaire engendrée par  $S_{\sigma(1)}, \dots, S_{\sigma(i+1)}$  pour  $0 \leq i \leq n-1$ ,  $\varepsilon$  étant la signature de  $\sigma$ ; s'il n'existe pas de telle permutation, on pose  $(\Delta, V) = 0$ ; on étend ce "produit scalaire" à  $\mathcal{L}$  par linéarité, et  $\Sigma$  est alors le module quotient de  $\mathcal{L}$  par le sous-module des combinaisons linéaires de simplexes "orthogonales" à tous les drapeaux. Pour les autres groupes classiques, la méthode est analogue, les simplexes et les drapeaux devant alors être restreints à ne contenir que des variétés linéaires "isotropes". *J. Dieudonné.*

**Belov, N. V.; and Tarhova, T. N.** Groups of color symmetry. Akad. Nauk SSSR. Kristallografiya 1 (1956), 4-13 (1 plate). (Russian)

The authors discuss the plane "black and white"-groups generated by plane motions and turning over of two dimensional figures, indicating by the two colors white and black the original and turned over positions. Instead of distinguishing the two positions by different colors one can obviously also split up the collocal planes into two parallel planes, planes at different levels. This leads the authors to consider many color groups in which more than two levels are considered. For a  $4_1$ -screw in space four colors are needed, a  $6_1$ -screw requires six colors. By making coincide the differently coloured planes with one plane, a correspondence of spatial groups and plane color mosaics is obtained. Several special cases are discussed in detail. The authors point out that classical theory would lead to eliminating the case of five and more than six colors. They show that also five and more colors would be desirable, e.g. corresponding to different centering of given structures. This would correspond to an increase of the number of fundamental plane Bravais lattices.

*E. M. Bruins (Amsterdam).*

**Belov, N. V.; and Tarhova, T. N.** On the group of the polyhedron of 48 faces. Akad. Nauk SSSR. Kristallografiya 1 (1956), 360-361. (Russian)

In order to simplify the composition of two operations of the 48-group [consisting of the group of the octahedron and its inversions] the authors divide the six faces of two cubes into eight triangles by the diagonals and the midparallels and establish a one to one correspondence between these 48 triangles and the operations of the 48-group. They indicate how to operate with the cubes to find the resultant operation of two operations of the group. They state that this method avoids the construction of the really too extensive Cayley-matrix of 2304 elements.

{The reviewer wishes to indicate that if one makes a list of the 48 permutations of the six vertices of an octa-

hedron under the operations of the group the law of composition of permutations shall produce the resulting operation in less time than is needed to find the corresponding triangles on the cubes.} *E. M. Bruins.*

**Tamura, Takayuki.** Indecomposable completely simple semigroups except groups. *Osaka Math. J.* 8 (1956), 35-42.

The author follows up a paper by D. Rees [*Proc. Cambridge Philos. Soc.* 36 (1940), 387-400; MR 2, 127] in which it is proved that a completely simple semigroup can be represented as a regular matrix semigroup. He investigates the structure of completely simple semigroups which have no non-trivial homomorphisms, first finding necessary and sufficient conditions for a regular matrix semigroup to have no non-trivial homomorphisms.

The main result is that if  $2 \leq l \leq m < 2^l$ , there exists a semigroup of the type discussed of order  $lm$ ; and it is isomorphic or anti-isomorphic to a certain regular matrix semigroup. The distinct (up to isomorphism) matrix semigroups are tabulated for  $l=2$  and  $l=3$ .

{T. S. Motzkin has independently obtained the same results.} *H. A. Thurston (Bristol).*

**Iséki, Kiyoshi.** On compact semi-groups. *Proc. Japan Acad.* 32 (1956), 221-224.

A semigroup  $S$  is called a homogroup if 1)  $S$  has an idempotent  $e$ , 2) for each  $a \in S$ , there exist  $a', a'' \in S$  such that  $aa' = e = a''a$ , and 3) for all  $a \in S$ ,  $ae = ea$ . Joining the long list of compact semigroup theorems is: Any compact commutative semigroup is a homogroup.

If for  $a, b \in S$ ,  $aS \cap bS \neq \emptyset \neq S_a \cap S_b$  we say  $S$  is reversible. Then a compact semigroup is a homogroup if and only if it is reversible.

For  $e$ , an idempotent of  $S$ , let  $K^{(e)} = \{a \in \overline{\gamma(a)}\}$ , where  $\overline{\gamma(a)} = \bigcup_{i=1}^{\infty} a^i$ ,  $\overline{\gamma(a)}$  = closure of  $\gamma(a)$ . We call a semigroup  $S$  strongly reversible in case for all  $a, b \in S$ , there are integers  $r, s, t$  such that  $(ab)^r = a^s b^t a^s$ . Then, if  $S$  is compact and strongly reversible, each  $K^{(e)}$  is a maximal semigroup containing  $e$  and  $S$  is the union of disjoint  $K^{(e)}$ . *B. Gelbaum (Minneapolis, Minn.).*

**Iséki, Kiyoshi.** Contribution to the theory of semi-groups. II. *Proc. Japan Acad.* 32 (1956), 225-227.

Let  $S$  be a compact semigroup. In its nonempty set of idempotents  $\{e_\alpha\}$ , let  $e_\alpha \leq e_\beta$  in case  $e_\alpha e_\beta = e_\alpha$ . Theorem: In a homogroup [see the preceding review] or compact abelian semigroup there is a unique least idempotent. Furthermore,  $\mathfrak{R} = \bigcap_{e_\alpha} S e_\alpha \neq \emptyset$ .

Let  $S$  be a topological homogroup,  $M = \{xe | x \in S\}$ , where  $e$  is an idempotent such that  $M$  is a group and two-sided ideal. G. Thierrin [*C. R. Acad. Sci. Paris* 234 (1952), 1519-1521; MR 13, 902] showed such an  $e$  exists.  $M$  is called the group ideal of  $S$ . A character of  $S$  is a character of  $M$  and a character of  $M$  can be extended to a character of  $S$ . If  $\hat{S}$  is the set of characters of  $S$ , then  $\hat{S}$  is a group and is the character group of  $M$ . *B. Gelbaum.*

**Iséki, Kiyoshi.** Contributions to the theory of semi-groups. III. *Proc. Japan Acad.* 32 (1956), 323-324.

A homogroup  $S$  is a semigroup having the following properties: 1)  $S$  contains an idempotent  $e$ , 2) for every  $x \in S$   $xe = ex$  holds, 3) for every  $x \in S$  there are two elements  $x', x''$  such that  $xx' = e = x''x$ . The author proves that any simple homogroup [i.e. a homogroup without proper two-sided ideals] is a group. [Misprint: p. 324 line 2 the word "group.. is to be replaced by "semigroup".] *St. Schwarz (Bratislava).*

**Iséki, Kiyoshi.** Contributions to the theory of semi-groups. IV. *Proc. Japan Acad.* 32 (1956), 430-435. [For notations and terminology see the two preceding reviews.] If  $\mathfrak{A}$  is a two-sided ideal in  $S$ , let

$$\bar{\mathfrak{A}} = \{a | a \in S, a^* \in \mathfrak{A}, \text{ some } n\}.$$

Then, if  $S$  is strongly reversible  $\bar{\mathfrak{A}}$  is a two-sided ideal.  $S$  is called periodic in case  $\{a^n | a \in S, n=1, 2, \dots\}$  is a finite set for all  $a$ . If  $\mathfrak{A}$  is a two-sided ideal, and  $e_\alpha$  are the idempotents of  $\mathfrak{A}$ , then  $\bar{\mathfrak{A}} = \bigcup_{\alpha} K^{(e_\alpha)}$ .

If  $\bar{\mathfrak{A}} = \mathfrak{A}$ ,  $\mathfrak{A}$  is called closed. A two-sided ideal  $P$  is called prime if  $ab \in P \Rightarrow a \in P$  or  $b \in P$ . For any family of prime ideals  $P_\alpha$ ,  $\bigcap P_\alpha$  is closed. If  $S$  is abelian periodic, each closed ideal is the intersection of prime ideals.

If  $S$  is strongly reversible, then the set  $E$  of all idempotents is an abelian semigroup, the ideals (prime ideals) of  $E$  and of  $S$  are in 1-1 correspondence.

Finally,  $S$  is a separating class of functions for the elements of distinct conjugate classes of an abelian periodic semigroup  $S$ . (For each  $K^{(e)}$  let  $G^{(e)}$  be the maximal subgroup of  $K^{(e)}$ . For  $a \in G^{(e)}$  the set  $T_a = \{x | xe_a = a\}$  is a conjugate class of  $S$ .)

*B. Gelbaum (Minneapolis, Minn.).*

**Schützenberger, Marcel Paul.** Sur deux représentations des demi groupes finis. *C. R. Acad. Sci. Paris* 243 (1956), 1385-1387.

Let  $\Sigma$  be a semigroup of mappings of a finite set into itself. Represent each mapping  $\sigma$  by an incidence matrix  $S_\sigma = (S_{xy})$ , where  $S_{xy} = 1$  if  $\sigma x = y$ , otherwise 0. Also consider  $S_{\sigma^{-1}} = (S_{xy}')$ , where  $S_{xy}' = 1$  if  $x = \sigma^{-1}y$ , otherwise 0. It is shown that there exists a module  $\mathcal{B}$  of matrices such that  $S_\sigma B = B S_{\sigma^{-1}}$  for each  $\sigma$  in  $\Sigma$  and each  $B$  in  $\mathcal{B}$ .

*H. Campaigne (Carlisle, Penn.).*

**★ Séminaire A. Châtelet et P. Dubreil de la Faculté des Sciences de Paris, 1953/1954. Partie complémentaire: Demi-groupes. 9 exposés par Croisot, R.; Riguet, J.; Teissier, Mlle. M.; Thibault, R.; et Thierrin, G. 2e tirage multigraphié. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. ii+84 pp.**

These are lectures from a seminar on algebra held in 1953 and 1954. It is quite a complete summary of the knowledge of semigroups up to date. The results presented are mostly those of the French, who have taken the lead in this subject. The significant work of others is not overlooked, however.

The French use the word "demi-groupe" for an associative multiplicative system, which in English is usually semigroup, and the word "semi-groupe" for a demi-groupe with the cancellation laws on both sides.

The first lecture is on general notions. These include all the basic ideas, including that of ideals. Among the various concepts is the important one of a "regular" equivalence, in which  $a = a'$ ,  $b = b'$  imply that  $ab = a'b'$ . The presence of this type of equivalence is necessary and sufficient for the existence of a semigroup which is the homomorphic image.

The second lecture concerns some special conditions under which a semigroup can be imbedded in a group. A later lecture takes up the general case.

The third lecture continues the subject of homomorphisms from semigroups to groups.

The fourth lecture takes up inversive semigroups and semigroups which are the unions of simple semigroups. A semigroup is "simple" if it has no proper ideals. An

ideal  $I$  is "semi-prime" if  $a^2$  in  $I$  implies that  $a$  is in  $I$ . A necessary and sufficient condition for a semigroup to be the union of simple semigroups is that every ideal be semi-prime. For a left ideal to be semi-prime means that for each  $a$  in the ideal there is an  $x$  in the semigroup so that  $a = xa^2$ ;  $x$  is a sort of inverse. By analogy G. Thierrin [C. R. Acad. Sci. Paris 232 (1951), 376-378; MR 12, 389] was led to consider those semigroups for which each  $a$  had an  $x$  so that  $a = axa$ ; these semigroups he called "inversive". If a semigroup  $D$  satisfies two of the following conditions then it satisfies the third also and is the union of groups. The three conditions are: (1)  $D$  is inversive; (2) the left ideals of  $D$  are semi-prime (also called left-inversive); (3) the right ideals of  $D$  are semi-prime. These semigroups have previously been studied by A. H. Clifford [Ann. of Math. (2) 42 (1941), 1037-1049; MR 3, 199] who showed that a necessary and sufficient condition for a semigroup to be a union of groups was that for each element  $x$  there should exist a relative identity  $e_x$  and a relative inverse  $x'$  so that  $e_x x = x = x e_x$  and  $xx' = e_x = x'x$ .

The fifth lecture is devoted to interior automorphisms. Despite the general absence of inverses, if  $x$  in  $D$  is such that  $x D = D x$  then  $a \rightarrow b$  if  $xa = bx$  is called an "interior automorphism"; the totality of them sometimes forms a group. This is further generalized to relations of the form  $xay = xbt$ , which, if it is an automorphism of  $D$ , is called a "generalized interior automorphism." These form a group in the presence of the cancellation law.

The sixth lecture is about completely simple semigroups.

The seventh is devoted to the recent work of the Russians. Among the new concepts is the one by V. V. Vagner of a "generalized inverse." The elements  $x$  and  $y$  are generalized inverses of each other if  $xyx = x$  and  $yx = y$ . In the presence of the cancellation law this inverse is unique. A commutative semigroup in which each element has a generalized inverse is called a "Vagner" semigroup. It is shown to be representable as a subset of the one-to-one mappings of a set  $E$ . If it exhausts the mappings of  $E$  it has only interior automorphisms.

The eighth lecture concerns equivalence relations, particularly those whose presence indicates that the semigroup is a group.

The ninth and last lecture is about the Lambek [Canad. J. Math. 3 (1951), 34-43; MR 12, 481] method of imbedding a semigroup in a group. This is a scheme for giving a geometric interpretation to Malcev's [Math. Ann.

113 (1937), 686-691] eight element condition, which says that for each set of eight elements  $a, b, c, d, x, y, u$ , and  $v$ ,  $ax = by$ ,  $au = bv$ , and  $cx = dy$  imply that  $cu = dv$ . This condition together with the cancellation law are necessary and sufficient conditions that the semigroup be imbeddable in a group. H. Campaigne (Carlisle, Penn.).

Weaver, Milo W. On the imbedding of a finite commutative semigroup of idempotents in a uniquely factorable semigroup. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 772-775.

A "uniquely factorable semigroup, UFS" is defined to be a finite commutative semigroup with an identity and with the three additional properties following. Since the divisors of the identity form a group  $U$  the homomorphic image  $S/U$  is defined, and in this semigroup (i) there exists at least one "prime"  $P$ , such that the only divisors of  $P$  are  $U$  and  $P$ , (ii) each element  $A$  in  $S/U$  has  $A^2 = A$ , (iii) each element  $A$  in  $S/U$  is expressible uniquely as a product of primes.

The main result is that each commutative semigroup in which each element is idempotent can be imbedded in a UFS. Clifford [Ann. of Math. (2) 39 (1938), 594-610] got a similar result. H. Campaigne (Carlisle, Pa.).

Bocconi, Domenico.  $P$ -gruppoide dei quozienti di un gruppoide con operatori. Rend. Sem. Mat. Univ. Padova 25 (1956), 176-195.

The author calls an algebra with one binary operation a groupoid (gruppoide); if the operation is associative the algebra is called a pseudo-group (pseudogruppo). The main theorem is concerned with embedding a groupoid-with-operators in a group-with-operators: Let  $P$  be a multiplicative pseudo-group, and  $M$  the set of elements  $a$  of  $P$  which have the property that  $x = y$  whenever either  $ax = ay$  or  $xa = ya$ . Suppose that  $P$  has a pseudo-group  $\bar{P}$  of quotients re  $M$ , and that  $G$  is a groupoid with  $P$  as operator domain. Then there is a groupoid  $\bar{G}$  of quotients re  $M$  of  $G$  if and only if  $u_1 = u_2$  whenever  $\alpha \in M$ ,  $u_1 \in G$ ,  $u_2 \in G$ , and  $\alpha u_1 = \alpha u_2$ . The theorem is applied to rings. H. A. Thurston (Bristol).

See also: Rogers, p. 271; Jaśkowski, p. 271; Kalman, p. 274; Le Veque, p. 283; Harish-Chandra, p. 318; Klein, p. 329.

## THEORY OF NUMBERS

### General Theory of Numbers

Cohn, Harvey. Some applied number theory. J. Soc. Indust. Appl. Math. 4 (1956), 152-167.

An expository paper which, by a discussion of such matters as random walk, flow of heat, and planetary perturbation, shows that the theory of numbers need not be "non-practical".

★ LeVeque, William Judson. Topics in number theory. Vols. 1 and 2. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1956. x+198 pp. and viii+270 pp. \$5.50 (vol. 1) and \$6.50 (vol. 2).

The first five chapters of volume I are concerned with the usual topics in elementary number theory with a more complete treatment of linear congruences than is usual in beginning texts. A brief introduction to the elementary number-theoretic functions follows, with

application to Erdős proof of Bertrand's conjecture. There is a chapter on sums of squares including the proof by A. Brauer and T. L. Reynolds that every positive integer is the sum of four squares. Hurwitz's theorem on rational approximation is proved by use of Farey series, and continued fractions are developed as a tool for rational approximations. Definite efforts are made through selected methods of proof and exercises to give the reader the flavor of number theory and prepare him for the more advanced material in the second volume. It is the second volume which makes the most distinctive contribution to the literature of number theory. In the first chapter it is good to see use made of matrices and group-theoretic concepts in the development of the modular group and the linear transformation and its application to the reduction of binary quadratic forms.

The second and third chapters of the second volume are concerned with an introduction to the theory of

algebraic numbers and some applications to cyclotomic fields and a proof of Kummer's theorem that if  $p$  is a regular prime,  $x^p + y^p + z^p = 0$  has no solution in rational integers  $x, y, z$  prime to  $p$ , with rather complete discussion of the allied cases of Fermat's last theorem. There is a section on  $x^3 + y^3 = y^3$  (where the last exponent is wrong in the table of contents). The proof is given of the Delaunay-Nagell theorem that  $x^3 + dy^3 = 1$  has at most one solution in integers  $x, y$  different from zero, and that, except for only a finite number of values of  $d$ , any solution  $x_1, y_1$  determines a fundamental unit  $x_1 + y_1 d^{\frac{1}{3}}$  of the field  $R(d^{\frac{1}{3}})$ . Chapter four is devoted to a proof of the algebraic form of the Thue-Siegel-Roth theorem: Let  $K$  be an algebraic number field of degree  $N$  and let  $\alpha$  be algebraic of degree  $n \geq 2$  over  $K$ . Then for each  $x > 2$ , the inequality  $|\alpha - \zeta| < (H(\zeta))^{-x}$  has only finitely many solutions  $\zeta$  in  $K$ , where  $H(\zeta)$  is the "height" of  $\zeta$ , that is the maximum absolute value of the coefficients of the irreducible primitive polynomial with rational integral coefficients of which  $\zeta$  is a zero. Though the methods seem to be essentially those of Roth, the result in this form appears not to have previously appeared in print. Applications are given to related Diophantine equations.

Chapter five deals with irrationality and transcendence, begins with proofs of the irrationality of  $e$  and  $\pi$  (the latter using Niven's proof) followed by Mahler's classification of transcendental numbers, the theorem of Schneider and its application to the Gelfond-Schneider theorem. Chapter six contains an elementary and a non-elementary proof of Dirichlet's theorem on primes in an arithmetic progression. Chapter seven is concerned with a proof of the prime number theorem.

The contributions of these two volumes are of two kinds: much of the material in the second volume does not occur elsewhere in an English textbook, and some is original with the author; he takes pains and space to keep the reader informed of the paths ahead and to try to show him why a certain procedure may give the results desired. Usually the methods by which a result is proved are more important than the result itself. The author is to be congratulated on writing a book with this in mind.

B. W. Jones (Boulder, Colo.).

Schinzel, André. Sur quelques propriétés des nombres  $3/n$  et  $4/n$ , où  $n$  est un nombre impair. *Mathesis* 65 (1956), 219-222.

Using several identities, two of which are attributed to Sierpiński, the author proves three theorems of which the following is typical: For every positive integer  $n > 1$  there exist distinct odd integers  $x, y, z$ , such that

$$\frac{3}{2n+1} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

L. Moser (Edmonton, Alta.).

Scherk, Peter. Bemerkungen zu einer Arbeit von Herrn Kanold. *J. Reine Angew. Math.* 196 (1956), 133-136.

Let  $a, d, f$  be positive integers such that  $(d, f) = 1$ . For every  $a$  let  $(Q(a))^2$  be the largest perfect square dividing  $a$ . It is proved that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\substack{a=1 \\ (a,f)=1 \\ Q(a) \geq d}}^n 1 = \prod_{p|f} \frac{p-1}{p} - \frac{6}{\pi^2} \prod_{p|f} \frac{p}{p+1} \cdot \sum_{b=1}^{d-1} \frac{1}{b^2}.$$

This generalizes a theorem of H.-J. Kanold [same *J.* 193 (1954), 250-252; MR 16, 569]. The writer also gives alternative proofs of some of Kanold's results.

I. Niven (Eugene, Ore.).

Fadini, Angelo. *Algebra di matrici diagonali ed algebra di Boole collegate a particolari classi modulo  $n$* . *Ricerca*, Napoli 6 (1955), no. 2, 20-26.

If  $n = p_1 \cdots p_r$  is the product of  $r$  distinct primes, then the ring of residue classes mod  $n$  forms a Boolean algebra under divisibility. Applications to elementary number theory.

G. Birkhoff (Cambridge, Mass.).

da Costa, Newton Carneiro Affonso. A note on Wilson's theorem. *Soc. Parana. Mat. Anuário* 2 (1955), 5-6. (Portuguese)

It is noted that Wilson's theorem may be transformed into: For  $p$  to be a prime it is necessary and sufficient that  $n \leq p$  should exist such that

$$(n-1)!(p-n)! \equiv (-1)^n \pmod{p}.$$

Various simple consequences are derived.

D. H. Lehmer (Berkeley, Calif.).

Langford, C. Dudley. Super magic squares. *Math. Gaz.* 40 (1956), 86-97.

A super magic square of order four is four-by-four matrix of elements 1, ..., 16, such that each row, column, diagonal, and set of corners adds up to 34, and such that these properties are preserved under cyclic permutations of rows or columns of order two. Various properties of these squares are discussed, particularly further transformations preserving the super magic property. Generalizations to eighth and ninth order squares are considered. The methods are largely experimental. R. J. Walker.

Tatarkiewicz, Krzysztof. Sur les puissances des entiers. *Ann. Univ. Mariae Curie-Skłodowska. Sect. A.* 8 (1954), 5-23 (1956). (Polish and Russian summaries)

The author considers the congruence

$$(1) \quad a^{\nu p + r} \equiv a^r \pmod{10^k}.$$

He designates by  $p(a, k)$  the smallest positive integer  $p$  such that, for some  $r$ , (1) holds for all positive integers  $\nu$ , and by  $r(a, k)$  the smallest of the non-negative integers  $r$  for  $p = p(a, k)$ . The existence of these numbers was shown by Sierpiński [*Ann. Soc. Polon. Math.* 23 (1950), 248-251, 252-258; MR 12, 674] who gave their values for  $a=2$ . The author gives a means of evaluating  $p$  and  $r$  for all  $a$  as follows. He defines  $\sigma(a, k) = p(a, k+1)/p(a, k)$  and proves that  $\sigma(a, k) \leq 10/(a, 10)$  for  $k=1, 2, 3, \dots$ , where there is a least integer  $N(a)$  such that the equality holds for all  $k \geq N(a)$ . A method is given for determining  $N(a)$  and the numbers  $p(a, i)$  for  $i \leq N(a)$ . {The last two exponents on page 13 should be  $Nn$  and  $N$  instead of  $10n$  and  $10$ ; a similar correction should be made two lines above.}

A table is given for  $2 \leq a \leq 32$  and  $a=51, 101, 251, 501, 751, 1251$  with accompanying values of  $p(a, 1)$ ,  $N(a)$ ,  $\sigma(k)$ ,  $r(k)$ .

B. W. Jones (Boulder, Colo.).

Dem'yanov, V. B. Pairs of quadratic forms over a complete field with discrete norm with a finite field of residue classes. *Izv. Akad. Nauk SSSR. Ser. Mat.* 20 (1956), 307-324. (Russian)

Let  $f_j(x_1, \dots, x_m)$  ( $1 \leq j \leq n$ ) be homogeneous forms in  $m$  variables of respective degrees  $r_j$  and coefficients in a field  $K$ . When  $K$  is the field of formal power series over a finite field it was shown by Lang [*Ann. of Math.* (2) 55 (1952), 373-390; MR 13, 726] that there is always a solution in  $K$ , other than the trivial one, of the simultaneous system  $f_j = 0$  provided that  $m > \sum r_j$ ; and it has been

conjectured that a similar result holds when  $K$  is a finite algebraic extension of a  $p$ -adic field. The author verifies this conjecture for the simplest simultaneous case, namely  $n=r_1=r_2=2$ ,  $m=9$ . The proof is technically completely elementary but uses an elaborate induction hypothesis and must consider separately many different possibilities. Let  $\mathfrak{p}$  denote the prime ideal of  $K$ . The greatest difficulty is to get from the solution of  $f_1=f_2=0 \pmod{\mathfrak{p}}$  provided by the work of Chevalley [Abh. Math. Sem. Hamburg. Univ. 11 (1935), 73-75] to a solution of  $f_1=f_2=0 \pmod{\mathfrak{p}^2}$ . The induction from  $\mathfrak{p}^N$  to  $\mathfrak{p}^{N+1}$  ( $N>1$ ), while still complicated, is rather simpler. A straightforward corollary is that if  $\mathfrak{p} \nmid 2$  and  $f(x_1, \dots, x_m)$ ,  $g(x_1, \dots, x_m)$  do not represent 0 simultaneously, where  $5 \leq m \leq 8$ , then there exist  $m-4$  numbers  $c$  of  $K$ , the quotient of no two of them being a square, such that  $f=g=c=0$  is simultaneously soluble. Frequent use is made of the lemma that if  $\sum a_{ij}x_i x_j$ ,  $\sum b_{ij}x_i x_j$  do not represent 0 simultaneously, where the  $a_{ij}$ ,  $b_{ij}$  are integers of  $K$  and the diagonal coefficients  $a_{ii}$ ,  $b_{ii}$  are divisible by  $\mathfrak{p}$ , then there exist integers  $\lambda, \mu$  of  $K$ , not both divisible by  $\mathfrak{p}$ , such that every  $\lambda a_{ij} + \mu b_{ij}$  is divisible by  $\mathfrak{p}$ .

*J. W. S. Cassels (Cambridge, England).*

**Ignat'eva, R. P.** Number of integral solutions of  $x+2y+3z+4u+5v+w=m$ . Kabardinskii Gos. Ped. Inst. Uč. Zap. 8 (1955), 53-59. (Russian)

**Ignat'eva, R. P.** Number of integral solutions of the equations  $x+2y+3z+4u=m$ ,  $x+2y+3z+4u+5v=m$ . Kabardinskii Gos. Ped. Inst. Uč. Zap. 8 (1955), 45-52. (Russian)

**Selmer, Ernst S.** On Cassels' conditions for rational solubility of the diophantine equation  $\eta^2=\xi^3-D$ . Arch. Math. Naturvid. 53 (1956), 115-137.

The problem of finding the number of generators (in the sense of Poincaré, Mordell and Weil) of the points with coefficients in the rational field  $K$  on the curve  $\eta^2=\xi^3-D$  leads to the consideration of equations

$$(*) \quad \xi - \delta = \mu x^2,$$

where  $\delta = D^{1/3}$ ,  $\mu$  is one of a fixed finite set of numbers of  $K(\delta)$ , and  $\xi \in K$ ,  $\alpha \in K(\delta)$  are variables. On expressing  $\mu, \alpha$  in terms of the basis 1,  $\delta, \delta^2$  of  $K(\delta)/K$  there results a pair of simultaneous quadratic equations in the coefficients of 1,  $\delta, \delta^2$  in  $\alpha$ . The author finds complete necessary and sufficient conditions for these equations to be simultaneously soluble to every modulus. These are sometimes neater than the partial set of necessary conditions used by the reviewer [Acta Math. 82 (1950), 243-273; MR 12, 11]. The author is also able to show that some of the reviewer's conditions are spurious, being implied by other conditions. A complete set of necessary conditions for the local solubility of (\*) is called for by the author's plausible conjecture about the "generators lost in second descents" [Math. Scand. 2 (1954), 49-54; MR 16, 14].

*J. W. S. Cassels (Cambridge, England).*

**Larsson, D. F.** Quelques inégalités de la théorie élémentaire des nombres. Mathesis 65 (1956), 205-210.

By considering corresponding congruences (mod 9) several Diophantine equations are shown to be insoluble. A typical result is: For  $m \not\equiv 0 \pmod{3}$ , the equation

$$\sum_{v=0}^{n+3m-1} (1+3\lambda)^v = b^c$$

has no solution in integers. *L. Moser (Edmonton, Alta.).*

**Vandiver, H. S.** Diophantine equations in certain rings. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 656-665.

In addition to several observations on the solution of Diophantine equations in rings, the following theorem is proved by considerations in the field generated by a primitive  $m$ th root of unity. The equation

$$da + k_1 p + (db + k_2 p)x^m = (d + k_3 p)y^m$$

with  $a, b$  and  $m$  fixed such that  $(a, b) = 1$  and  $m$  is an odd prime, is impossible for an infinity of integers  $c$  such that  $cm+1 \equiv 0 \pmod{p}$ ,  $d \not\equiv 0 \pmod{p}$ , and none of  $a, a+b, a(m+b^m)/(a+b)$  is an  $m$ th power. *I. Niven.*

**Leicht, J.** Über ZPE-Ringe in der algebraischen Geometrie. Monatsh. Math. 60 (1956), 214-222.

Contre-exemples à des assertions fausses de G. Kantz [Monatsh. Math. 59 (1955), 104-110; MR 17, 341]. Les assertions fausses sont du type suivant: tout anneau factoriel est un anneau de polynômes sur un anneau principal. Les contre-exemples sont des anneaux locaux de nature très simple, et des anneaux de coordonnées d'hypersurfaces affines non réglées dont toutes les sous-variétés de codimension 1 sont des intersections complètes.

*P. Samuel (Clermont-Ferrand).*

**Carlitz, L.** An application of a theorem of Stickelberger. Simon Stevin 31 (1956), 27-30.

Let  $f(x) = a_N x^N + a_{N-1} x^{N-1} + \dots + a_0$  be a polynomial with rational integral coefficients and with discriminant  $D \neq 0$ . Let  $p$  be an odd prime, not dividing  $D$ . It was shown by Stickelberger that the Legendre symbol  $(D/p)$  could be evaluated in terms of the number of irreducible factors of  $f(x) \pmod{p}$ . The author extends this result to finite fields of order  $q$ , and uses it to evaluate the number of solutions of the equation  $D(x_0, x_2, \dots, x_N) = y^2$  in  $\text{GF}(q)$ , where  $D = D(a_0, a_1, \dots, a_N)$  is the discriminant of  $f(x)$ . *R. Bellman (Santa Monica, Calif.).*

See also: Ingleton, p. 273; Skolem, p. 275; Skopin, p. 276; Gluškov, p. 280; Weaver, p. 283; Mordell p. 287; Oikawa, p. 290; Eichler, p. 297; Goldberg and Varga, p. 304; Châtelet, p. 334.

## Analytic Theory of Numbers

**Cohen, Eckford.** An extension of Ramanujan's sum. III. Connections with totient functions. Duke Math. J. 23 (1956), 623-630.

In this paper the author concludes his study of the properties of the extended Ramanujan sum  $c_k(n, r)$ , defined for integers  $n, r \geq 1, k \geq 1$  by  $\sum \exp(2\pi i n x / r^k)$ , the summation ranging over a  $k$ -reduced residue system (mod  $r$ ). He first obtains a generalized Hölder formula for  $c_k(n, r)$  which includes the relation  $c_1(n, r) = \phi(r)\mu(d)/\phi(d)$ , where  $d = r/(n, r)$ ,  $\phi(d)$  is the Euler function, and  $\mu(d)$  is the Möbius function. Next he derives several equivalent forms of the orthogonality relation for  $c_k(n, r)$  established in part II [same J. 22 (1955), 543-550; MR 17, 238]. As an application he deduces an explicit formula for the number of solutions of the linear congruence  $n \equiv \sum_{i=1}^k x_i \pmod{r^k}$ , wherein each  $x_i$  has no  $k$ th power divisor  $\neq 1$  in common with  $r^k$ . Finally he obtains a Dirichlet series expansion in terms of  $c_k(n, r)$  for the generalized Euler function  $\phi_k(r)$  defined as the number of integers in a  $k$ -reduced residue system (mod  $r$ ). *A. L. Whiteman.*

**Carlitz, L.** A note on Gauss' sum. *Proc. Amer. Math. Soc.* 7 (1956), 910-911.

A simple algebraic evaluation of the Gaussian sum  $S = \sum_{r=1}^{p-1} \exp(2\pi i r^2/p)$ , for the case of an odd prime  $p$ , is given. This is accomplished by considering the determinant  $D = |e^{2\pi i r s/p}|$ , where  $e = \exp(2\pi i/p)$ ,  $g$  = a primitive root, modulo  $p$ , and  $r, s = 0, 1, \dots, p-2$ . The value of this determinant is known. On the other hand  $D$  is a circulant and can be written as a product of factors one of which is  $S$ , another is trivial, and the rest are paired off in such a way that the products of each pair are known. Equating the two results gives the value of  $S$ .

H. W. Brinkmann (Swarthmore, Pa.).

**Bellman, Richard.** On a class of functional equations of modular type. *Proc. Nat. Acad. Sci. U.S.A.* 42 (1956), 626-629.

The Voronoi functions

$$V_a(x, y) = \int_0^\infty \exp(-\pi x^2 s - \pi y^2 s^{-1}) s^{-a} ds$$

can be used for building the generalized theta functions

$$f(x, y; u, v; t_1, t_2) = \sum_{m, n=-\infty}^{\infty} V_a((x+m)t_1^{1/2}, (y+n)t_2^{1/2}) e^{2\pi i(mu+nv)}$$

with the functional equation

$$f(x, y; u, v; t_1, t_2) = t_1^{-1/2} t_2^{-1/2} e^{-2\pi i(xu+yv)} \cdot f(v, u; -y, -x; t_2^{-1}, t_1^{-1}).$$

As  $V_a(x, y) = x^{2a-2} V_a(|xy|)$ , the function  $f$  can be written in terms of divisor functions. The author indicates several divisor functions. The author indicates several generalizations: (i) to generalized Voronoi functions, (ii) to sums related to totally real algebraic number fields, (iii) to matric fields. He points out that the analogue of  $V_a(x, y)$  in finite fields is given by the Kloosterman sums.

N. G. de Bruijn (Amsterdam).

**Grosswald, Emil.** The order of the zeta function in the critical strip. *Duke Math. J.* 23 (1956), 621-622.

In addition to correcting an oversight in the author's previous paper [*Duke Math. J.* 23 (1956) 41-44; MR 17, 588], an inequality for the order  $\mu(\sigma)$  of  $\zeta(\sigma+it)$  is given. A special case of this is: (\*)  $\mu(1-l/(2^l-2)) \leq a/(2^l-2)$ ,  $a=1.0276$ , if  $l \geq 4$  is a real number (rather than an integer). If  $l$  is not too large, the author's result is superior to an earlier result of van der Corput and Koksma [*Ann. Fac. Sci. Univ. Toulouse* (3) 22 (1930), 1-39]. Both the latter and the former are obtained from a still earlier result of van der Corput [*Math. Z.* 29 (1928), 397-426] of the same form (\*), but with  $a=1$ , by using the convexity of  $\mu(\sigma)$ .

L. Schoenfeld (East Pittsburgh, Pa.).

★ **Teghem, J.** Sur des applications de certaines estimations de sommes trigonométriques. *Colloque sur la Théorie des Nombres*, Bruxelles, 1955, pp. 183-204. Georges Thone, Liège; Masson and Cie, Paris, 1956.

Dans cet exposé l'auteur étudie les améliorations qui peuvent résulter de l'application des résultats d'estimation de  $|\sum_{r=1}^{p-1} \exp(2\pi i r^2/x)|$  obtenus dans les vingt dernières années par Vinogradov, van der Corput, Hua, Titchmarsh, Tchudakoff et autres. Dans les sommes trigonométriques  $\sum_{r=1}^{p-1} \exp(2\pi i r^2/x)$ ,  $f(x)$  est un polynôme à coefficients réels de degré  $k$ , ou une fonction réelle proche dans un certain sens, d'un polynôme de degré  $k$ . Les limitations supérieures non triviales de  $|\sum_{r=1}^{p-1} \exp(2\pi i r^2/x)|$

s'obtiennent sous la forme  $c(k)X^{1-\Lambda(k)}$ . Les auteurs susdits, tout comme Vinogradov lui-même dans ses premiers recherches, ont porté tous leurs efforts sur l'accroissement, dans toute la mesure possible, du nombre  $\Lambda(k)$ , sans se préoccuper de  $c(k)$ . Or, si l'ordre de grandeur de  $c(k)$  ne présente effectivement aucun intérêt dans les problèmes de la théorie des nombres, où  $k$  est fixe lorsque  $X \rightarrow \infty$  (p.e. le problème de Waring; systèmes d'un nombre fixé d'inéquations diophantiennes), il joue au contraire un très grand rôle dans les problèmes où  $k$  tend vers l'infini avec  $X$  (p.e. dans les recherches sur l'estimation de l'ordre de grandeur de  $\zeta(s)$  dans le voisinage de  $\sigma=1$ ). Il faut même alors que l'ordre de grandeur de  $c(k)$  ne compromette pas l'amélioration apportée à  $\Lambda(k)$ . Les estimations récentes les plus exactes de  $c(k)$  sont de la forme  $c(k) = c \exp\{ak \log^2 k\}$  ( $c$  = constante positive,  $a=32$  (Titchmarsh),  $a=18$  (Flett)). L'auteur exprime l'opinion que l'abaissement de l'ordre de grandeur de  $c(k)$  constitue un problème difficile. Par rapport à la limitation supérieure de  $\pi(x) - li x$  l'auteur montre que le résultat de Tchudakoff

$$\pi(x) - li x = O(x \exp\{-A(\log x)^{5/9-\epsilon}\})$$

[C. R. (Dokl.) Acad. Sci. URSS (N.S.) 21 (1938), 421-422] n'est nullement amélioré en appliquant les derniers résultats de Vinogradov. Il remarque enfin qu'un abaissement de l'ordre de grandeur de  $c(k)$  à  $k^\epsilon$  donnerait lieu au résultat amélioré

$$\pi(x) - li x = O(x \exp\{-A(\log x)^{3/5-\epsilon}\}).$$

Vinogradov a annoncé à deux reprises en 1942 et 1945 la démonstration par Tchudakoff d'un tel résultat. Mais aucune démonstration n'a paru, à la connaissance de l'auteur. Le résultat annoncé par Vinogradov est encore très éloigné du résultat  $\pi(x) - li x = O(\exp\{-A \log x\})$ , valables si l'hypothèse de Riemann est vraie. Il est plus près du résultat de de la Vallée Poussin  $\pi(x) - li x = O(x \exp\{-A(\log x)^{1/2}\})$  [Mém. Couronnés Autres Mém. Acad. Roy. Sci. Lett. Beaux-Arts Belg. Coll. in 8° 59 (1899), no. 1].

S. C. van Veen.

### Theory of Algebraic Numbers

**Selmer, Ernst S.** Tables for the purely cubic field  $K(\sqrt[3]{m})$ . *Avh. Norske Vid. Akad. Oslo. I.* 1955, no. 5, 38 pp. (1956).

Two tables are given for the cubic field  $K = K(\theta)$ , where  $\theta$  is the real cube root of the cube-free integer  $m$ .

The first table is for  $1 < m \leq 50$  and gives for each such  $m$  the factorization of rational primes  $N \leq 50$  which actually split in  $K$ . More precisely for each such  $N$  is given an integer  $\alpha = a + b\theta + c\theta^2$  of  $K$  whose norm is  $N$ . The simplest choice of  $a, b, c$  is made in each case. Since  $(a, b, c)$  or  $(3a, 3b, 3c)$  are rational integers we obtain solutions  $x, y, z$  of the Pellian cubic

$$x^3 + my^3 + m^2z^3 - 3mxyz = M$$

for  $M = N$  or  $27N$ . The solution  $d$  of the congruence

$$d^3 \equiv m \pmod{N}$$

is also given. Auxiliary data pertaining to  $K$  itself is given also. This includes the class number  $h$ , the basic unit  $\epsilon$ ,  $0 < \epsilon < 1$ , its reciprocal  $\epsilon^{-1}$ , the factorizations of the prime factors of  $3m$  and the generators of the ideal class group when  $h > 1$ . All such groups are cyclic for  $m \leq 50$ .

The second table gives the class number and basic unit for  $50 < m \leq 100$ . This extends a recent table of Cassels for  $m \leq 50$  [Acta Math. 82 (1950), 243-273; 84 (1951), 299; MR 12, 11, 481]. *D. H. Lehmer* (Berkeley, Calif.).

**Lekkerkerker, C. G.** On the Minkowski-Hlawka theorem. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 426-434.

The author shows that if  $K$  is a symmetric convex body of volume  $V$  in  $n$  dimensions then for  $n \geq 5$  there exists a  $K$ -admissible lattice of determinant not exceeding  $V/c$ , where  $c = 4.921 \dots$  is the solution of  $c \log c = 2(c-1)$ . Further, for  $n \geq 6$  the basis may be chosen to lie in the cube  $|x_i| \leq 2.13(V/\kappa_n)^{1/n}$ , where  $\kappa_n$  is the volume of the unit sphere, after a suitable rotation of the coordinate system. The second result remains valid for  $n=2, 3, 4, 5$  provided that 3, 3.82, 4.41, 4.80 respectively are substituted for  $c$ . The first result improves one of Davenport and Rogers [Duke Math. J. 14 (1947), 367-375; MR 9, 11]; the proofs here are an elaboration of theirs and involve some detailed computation. [For results for general starbodies which are sometimes stronger see Rogers, Proc. London Math. Soc. (3) 6 (1956), 305-320; MR 18, 21.] *J. W. S. Cassels* (Cambridge, Mass.).

See also: Skopin, p. 276; Masuda, p. 280; Vandiver, p. 285; Bellman, p. 286; Mordell, p. 287; Eichler, p. 297.

### Geometry of Numbers

★ **Poitou, G.** Approximations diophantiennes et groupe modulaire. Séminaire A. Châtelet et P. Dubreil de la Faculté des Sciences de Paris, 1953/1954. Algèbre et théorie des nombres. 2e tirage multigraphié, pp. 7-01 - 7-06. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956.

This paper [also published in Séminaire d'algèbre et de

théorie des nombres dirigé par A. Châtelet et P. Dubreil, 1953/1954, Fac. Sci. Paris, 1954; MR 16, 1082] is an exposition of the applications of a geometric method of Speiser [see Koksma, Diophantische Approximationen, Springer, Berlin, 1936, p. 43]. The author shows among other things that this method easily leads to lower bounds for the Hurwitz constants of the quadratic fields  $R(\sqrt{-m})$  ( $m=1, 2, 3, 7, 11, 19$ ), which do not differ much from the known exact values of these constants.

*C. G. Lekkerkerker* (Amsterdam).

**Mordell, L. J.** Diophantine inequalities in complex quadratic fields. Publ. Math. Debrecen 4 (1956), 242-255.

The paper deals with the problem of finding the best possible constant  $k=k(D)$  such that, if  $a, b, c, d$  are complex numbers and  $|ad-bc|=1$ , the inequality

$$(1) \quad |(ax+by)(cx+dy)| \leq k$$

can be satisfied by complex numbers  $x, y$  of a prescribed form, e.g. by integers in  $K(i\sqrt{D})$ . Such integers are of the form

$$m+n\left(\frac{D'+i\sqrt{D'}}{2}\right)=m+n\omega,$$

where  $-D'$  is the discriminant of the field  $K(i\sqrt{D})$  and where  $m, n$  are rational integers. The general problem considered is to investigate (1) for  $x=u_1+v_1\omega, y=u^2+v_2\omega$ , where  $u$  and  $v$  denote real numbers with assigned residues (mod 1). Several applications.

*J. F. Koksma* (Amsterdam).

See also: Teghem, p. 286.

## ANALYSIS

### Functions of Real Variables

★ **Schwartz, L.** Compléments de calcul intégral: séries et intégrales. Les cours de Sorbonne. Méthodes mathématiques de la physique, I, 64 pp. Paris, 1955.

These are mimeographed notes of lectures, in 8 parts, bound in separate pamphlets (Part V was not available for review). Graduate students will find these to be valuable reading. The material is fundamental for analysis, the presentation is clear and interesting, the examples have depth.

Part I. Chapt. I: Elementary properties of  $\sum u_i$ ;  $i$  ranges over an arbitrary collection of indices and the  $u_i$  are real or complex numbers. Chapt. II: Properties of the Lebesgue Integral (for one or several variables), particularly under change of variables, are sketched. The integral is described as a linear functional, measure for a set is then derived. However, measurable and summable functions are not defined precisely, and proofs are omitted in this chapter. Chapt. II-bis: discusses improper Riemann integrals, trigonometric integrals, principal Cauchy values. Chapt. III: Series of functions, and representation of functions by series and by integrals are discussed, together with such questions as derivation of a series term by term and derivation of an integral (under the integral sign).

*I. Halperin* (Kingston, Ont.).

★ **Schwartz, L.** Théorie élémentaire des distributions. Les cours de Sorbonne. Méthodes mathématiques de la physique, II, 36 pp. Paris, 1955.

The reader who finds the author's "Théorie des distributions" [2 vols., Hermann, Paris, 1950, 1951; MR 12, 31, 833] too difficult will find here a gentle introduction to the theory of distributions (created by the author himself) with many illustrative examples.

*I. Halperin.*

★ **Schwartz, L.** Convolution. Les cours de Sorbonne. Méthodes mathématiques de la physique, III, 40 pp. Paris, 1955.

Direct and convolution products of distributions are studied. The examples are particularly valuable. Attention is called to particular families of distributions for which convolution is always defined. These algebras of convolutions have applications in physics, the solving of linear differential equations, Heaviside calculus, Integral equations of Volterra, etc.

*I. Halperin.*

★ **Schwartz, L.** Séries de Fourier. Les cours de Sorbonne. Méthodes mathématiques de la physique, IV, 32 pp. Paris, 1955.

Fourier series for periodic functions and distributions; point-wise and distribution-wise convergence of the series; Hilbert space of square summable functions and

its completeness (proof: via convolution of distributions), Bernoulli polynomials.

*I. Halperin.*

★ **Schwartz, L. Transformations de Laplace.** Les cours de Sorbonne. Méthodes mathématiques de la physique, VI, 25 pp. Paris, 1955.

The usual calculus of Laplace transforms is made more coherent here by working within the framework of distributions. The misleading formula: the function 1 has Laplace transform  $p^{-1}$  is replaced by:  $Y(t)$  has Laplace transform  $p^{-1}$  so that after derivation, a correct result:  $\delta$  has Laplace transform 1 is obtained. Convolutions are used to derive many interesting Laplace transforms.

*I. Halperin (Kingston, Ont.).*

★ **Schwartz, L. Équations des cordes vibrantes.** Les cours de Sorbonne. Méthodes mathématiques de la physique, VII, 45 pp. Paris, 1956.

Vibrations of a stretched string, or in a column of gas or liquid, longitudinal vibrations in a solid, problem of Cauchy.

*I. Halperin (Kingston, Ont.).*

★ **Schwartz, L. Fonctions spéciales.** Les cours de Sorbonne. Méthodes mathématiques de la physique, VIII, 24 pp. Paris, 1956.

This pamphlet is along classical lines (i.e. without use of distributions): The  $\Gamma$ -function, Gauss integrals, Beta-function, for one or several dimensions; Stirling's formula, infinite products (particularly for expressing  $(\Gamma(x))^{-1}$  and  $\sin x$ ), and Bernoulli numbers, are treated. The inequality  $\Gamma''T - (\Gamma')^2 > 0$  is proved.

*I. Halperin.*

**Brîș, N. I. Sequences distributed uniformly in the mean on a segment.** Grodzenskii Gos. Ped. Inst. Uč. Zap. 1 (1955), 21–23. (Russian)

**Livingston, A. E.; and Lorch, Lee. The zeros of certain sine-like integrals.** Proc. Amer. Math. Soc. 7 (1956), 813–816.

If  $f(t) \geq 0$  in  $(0, 1)$ ,  $f(t) \not\equiv 0$  on any subinterval,

$$(-1)^n f(t+n) = f(t),$$

and  $f(t)/t \in L(0, 1)$ , the authors show that  $\int_0^\infty t^{-1} f(t) dt$  has zeros  $z_n$  such that  $z_n - n$  decreases to the limit  $C$  defined by  $2 \int_0^C f(t) dt = \int_0^1 f(t) dt$ ,  $0 < C < 1$ . The key to the proof is the observation that  $t \sum_{k=0}^\infty (-1)^k / (t+k)$  decreases for  $t > 0$ . The authors remark in this connection that (with  $\varphi = \Gamma'/\Gamma$ ) the function  $t^\delta (\varphi(t+\frac{1}{2}) - \varphi(t))$  is completely monotonic on  $(0, \infty)$  if and only if  $\delta \leq 1$ ; if  $1 < \delta < 2$  it increases for  $0 < t \leq (\delta-1)/(2-\delta)$ , and if  $\delta \geq 2$  it increases for all positive  $t$ .

*R. P. Boas, Jr. (Evanston, Ill.).*

**Alaci, V. Contribution concernant les "fonctions quasi homogènes".** Acad. R. P. Romîne. Baza Cerc. Ști. Timișoara. Stud. Cerc. Ști. Ser. I. 2 (1955), 13–20. (Romanian. Russian and French summaries)

The function  $f(x, y, z)$  is called "almost homogeneous" provided that there exist differentiable functions  $\varphi_1(t)$ ,  $\varphi_2(t)$ ,  $\varphi_3(t)$ ,  $\phi(t)$ , such that

$$(*) \quad f(x + \varphi_1(t), y + \varphi_2(t), z + \varphi_3(t)) = \phi(t) f(x, y, z).$$

The equivalence of  $(*)$  to

$$(**) \quad \varphi_1 \frac{\partial f}{\partial x} + \varphi_2 \frac{\partial f}{\partial y} + \varphi_3 \frac{\partial f}{\partial z} = f \cdot \log \phi$$

has been shown [Alaci, Com. Acad. R. P. Române 2 (1952), 113–115; MR 17, 45]. In the present paper it is

shown that if  $f_k(x, y, z)$  satisfy  $(**)$ , so does  $\sum_{k=1}^n A_k f_k$ ,  $A_k$  constants, while  $\prod_{k=1}^n f_k$  satisfies  $(**)$  with  $\phi^n$  instead of  $\phi$ . More generally, if  $F_p(x, y, z)$  is a polynomial in the  $f_k$ , homogeneous of degree  $p$ , then  $F_p$  satisfies  $(**)$  with  $\phi_p$  instead of  $\phi$ . The paper concludes with the presentation of several examples of almost homogeneous functions.

*E. Grosswald (Philadelphia, Pa.).*

**Matschinski, Matthias. Introduction des moyennes dans les équations de la mécanique et principe de Saint-Venant.** C. R. Acad. Sci. Paris 243 (1956), 1273–1276.

**Krumbach, Günther. Über den symbolischen Kalkül mit  $n$  Variablen.** I. Ann. Univ. Sarav. 4 (1955), 238–260 (1956).

In his "Hyperbolic differential equations" [Inst. Advanced Study, Princeton, 1953; MR 16, 139] Leray introduces a set  $F(\Delta)$  of functions which possess a Laplace transform. The derivatives of these functions are defined with the aid of the Laplace transform. The present author shows that there exists a locally convex topological vector space  $F_\infty(\Delta)$  of infinitely differentiable functions such that the set  $F(\Delta)$  may be identified with a subset of the set  $\mathfrak{C}$  of all continuous linear functionals on  $F_\infty(\Delta)$ . The derivatives of the functions of  $F(\Delta)$  may then be defined as derivatives in the sense of Schwartz's distributions. Let  $\mathfrak{A}$  be the set of all continuous linear operators on  $F_\infty(\Delta)$  which commute with translation. The author shows that there exists a one to one mapping of  $\mathfrak{C}$  onto  $\mathfrak{A}$ . The operators of  $\mathfrak{A}$  commute with differentiation. The multiplicative product of an operator of  $\mathfrak{A}$  with an arbitrary continuous linear operator on  $F_\infty(\Delta)$  belongs to  $\mathfrak{A}$ . Let  $\Phi_\infty(\Delta)$  denote the set of the Laplace transforms of the functions of  $F_\infty(\Delta)$ . The Laplace transform  $B'$  of a continuous linear operator  $B$  on  $F_\infty(\Delta)$  is defined by the equation

$$B' \varphi = L B L^{-1} \varphi, \quad \varphi \in \Phi_\infty(\Delta);$$

it is a continuous linear operator on  $\Phi_\infty(\Delta)$ . The author shows that the Laplace transform of an operator of  $\mathfrak{A}$  is obtained by multiplication by an analytic function.

*J. Korevaar (Madison, Wis.).*

See also: Džvarševili, p. 297; Bahvalov, p. 314; Young, p. 316; Nikol'skii, p. 321.

### Measure, Integration

**Suster, H. S. On a certain singular integral**

$$f_n(x) = \lambda_n \int_{-n}^n f(x+t) \frac{\varphi(nt)}{\varphi(t)} dt.$$

Grodzenskii Gos. Ped. Inst. Uč. Zap. 1 (1955), 79–89. (Russian)

**Šadrina, N. Ya. A type of singular integrals.** Grodzenskii Gos. Ped. Inst. Uč. Zap. 1 (1955), 73–78. (Russian)

**Choquet, Gustave. Unité des représentations intégrales au moyen de points extrémaux dans les cônes convexes réticulés.** C. R. Acad. Sci. Paris 243 (1956), 555–557.

$X$  is a locally convex vector space,  $C$  a cone in  $X$  such that for an affine space  $ACX$ ,  $B = C \cap A$  is compact and generates  $C$ . Suppose the order relation defined in  $X$  by  $C$  is a lattice ordering. Let  $M$  be the set of positive Radon

measures on  $B$ . For every  $\mu \in M$ ,  $x_\mu = \int x d\mu$  is called the resultant of  $\mu$ . Let  $E$  be the set of extreme points of  $B$ . The main theorem is that, for every  $x \in C$ , there is a measure  $\mu \in M$ , whose support is  $E$ , such that  $x = x_\mu$ .  
C. Goffman (Norman, Calif.).

**Bartle, R. G.** A general bilinear vector integral. *Studia Math.* 15 (1956), 337-352.

The paper develops an integral in which the product of a point function and a measure function is replaced by a continuous bilinear operation on the product space  $X \times Y$  of two linear normed complete spaces to a linear normed complete space  $Z$ . Similar generalizations have been previously made by Gowurin [*Fund. Math.* 27 (1936), 254-268], Price [*Trans. Amer. Math. Soc.* 47 (1940), 1-50; MR 1, 239], and Rickart [*ibid.* 52 (1942), 498-521; MR 4, 162]. If  $x(s)$  is a function on an abstract set  $S$  to  $X$  and  $\mu(E)$  an additive set function on  $\mathcal{S}$ , a field of subsets of  $S$ , to  $Y$ , then a basic integral for a step function  $x(s) = \sum_{i=1}^n x_i \chi_{E_i}(s)$  on any set  $E$  of  $\mathcal{S}$  is defined in the customary way:

$$\int_E x(s) \mu(ds) = \sum_{i=1}^n x_i \mu(E_i \cap E).$$

If  $x(s)$  is the limit in measure of a sequence of step function  $x_n(s)$  then the integration process can be extended to  $x(s)$  provided the sequence  $\lambda_n(E) = \int_E x_n(s) \mu(ds)$  are uniformly absolutely continuous relative to  $\|E\|$  (where  $\|E\| = \sup \{ \sum_{i=1}^n x_i \mu(E_i) \}$  for all finite subdivisions of  $E$  and all  $\|x_i\| \leq 1$ ) and if for every  $G > 0$ , there exists a set  $E_\epsilon$  in  $\mathcal{S}$ , for which  $\|E_\epsilon\| < \infty$  such that  $G$  in  $\mathcal{S}$  with  $G \subset E_\epsilon$  implies  $\|\lambda_n(G)\| < \epsilon$ . The resulting integral has many of the usual properties of Lebesgue integration including a Vitali theorem valid under convergence in measure of the functions  $f_n(s)$ , uniform absolute continuity of the integrals  $\int_E f_n(s) \mu(ds)$  relative to  $\|E\|$  and equicontinuity of the integrals relative to  $\|E\|$ . The dominated or bounded convergence theorems of Lebesgue must be slightly altered. If the function  $\mu$  is countably additive in the sense that for  $E_n$  disjoint  $\mu(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n)$ , the series on the right being unconditionally convergent, then the integrals give rise to a countably additive set function on  $\mathcal{S}$  to  $Z$ .

T. H. Hildebrandt (Ann Arbor, Mich.).

See also: Schwartz, p. 287; Džvarševili, p. 297; Helgason, p. 319; Aronszajn and Smith, p. 319; Michael, p. 325; Rényi, p. 339; Császár, p. 340.

### Functions of Complex Variables

**Polak, A. I.** On the extension of covering theorems in the theory of analytic functions to sufficiently broad classes of continuous mappings. *Dokl. Akad. Nauk SSSR (N.S.)* 106 (1956), 970-972. (Russian)

The author proves the theorem: Let  $\mathcal{F}$  be a family of continuous open mappings  $f$  of the disk  $|z| < 1$  in the complex  $z$ -plane into the complex  $w$ -plane. Let the family  $\mathcal{F}$  be compact, let  $f(0) = 0$  for every  $f$  in  $\mathcal{F}$ , and, for every function  $F$  in  $\mathcal{F}$  which is limit of a sequence  $f_n(z)$  converging uniformly for  $|z| < r < 1$ , let there be some circumference  $|z| = \delta < r$  on which  $F \neq 0$ . Then there exists  $\rho > 0$  such that every  $f$  in  $\mathcal{F}$  maps the disk  $|z| < 1$  on a set covering the disk  $|w| < \rho$ . Various generalizations are considered and a related theorem on normal families of analytic functions is proved.  
W. Kaplan.

**Štěpánek, Jiří.** Expansion of an analytic function into a "Taylor" series with variable center. *Časopis Pěst. Mat.* 81 (1956), 38-42. (Czech)

The author considers a Taylor series

$$f(z) = \sum f^{(n)}(\zeta)(z-\zeta)^n/n!$$

with variable center  $\zeta$ ,  $\zeta = (1-t)z_0 + tz_1$ ,  $z_0, t$  fixed. As such it is no more a power series and its domain of convergence is discussed.  
František Wolf.

**Barocio, Samuel.** Singularities of analytical differential systems in the plane. *Proc. Nat. Acad. Sci. U.S.A.* 42 (1956), 765-766.

The author has considered systems of the form  $\dot{x} = [x, y]_p$ ,  $\dot{y} = [x, y]_q$ , where  $[x, y]_r$  denotes a power series in  $x, y$  convergent near the origin and beginning with terms of degree  $\geq r$ . It is supposed that the origin is an isolated singular point. The author reports that he has obtained "a complete classification of the 'local phase-portrait' ... modulo 'fans'." No details are given but a complete exposition of the results is planned for later publication.  
C. E. Langenhop (Ames, Iowa).

**Kaufmann, I.** Au sujet de l'ensemble singulier parfait et totalement discontinu d'une fonction analytique uniforme partout continue. *Acad. R. P. Romine. Baza Cerc. Ști. Timișoara. Stud. Cerc. Ști. Ser. I.* 2 (1955), 41-44. (Romanian. Russian and French summaries)

Let  $P$  be a subset of a set  $M$ , such that  $P$  and  $M-P$  are both closed; then  $P$  is called a portion of  $M$ . Let  $Z = f(z)$  be a uniform analytic function, with the following properties: (i) The set  $M$  of its singular points is perfect and totally disconnected and (ii)  $f(z)$  is continuous on  $M$ . Let  $C$  be the set of values of  $Z$  and denote by  $G$  the inverse image of  $F(C)$ , the boundary of  $C$ . Then  $G \subset M$  and  $G$  does not contain any portion of  $M$ . This theorem has been previously proven by the author [*An. Acad. R. P. Române. Ser. Mat. Fiz. Chim.* 3 (1950), 119-138] under the assumption that the set of components of  $F(C)$  is countable. In the present paper, the author proves the theorem without that assumption and gives an example showing that the set of components of  $F(C)$  might not be countable.

E. Grosswald (Philadelphia, Pa.).

**Morgenstern, Dietrich.** Begründung des alternierenden Verfahrens durch Orthogonalprojektion. *Z. Angew. Math. Mech.* 36 (1956), 255-256.

The alternating method of Schwarz enables one to solve the Dirichlet problem for the equation  $\Delta u = 0$  in the union  $G = G_1 \cup G_2$  of two domains if it can be solved in each of the  $G_i$ . The method also applies to certain other differential equations, namely those whose solutions satisfy the principle of the maximum. In the present paper, which is in effect an abstract containing no proofs or details, the author states that the method may also be extended to a wider class of differential equations, including the equation  $\Delta \Delta u = 0$ . The method consists of regarding the solution of the Dirichlet problem in  $G_i$  as an orthogonal projection  $P_i$  in Hilbert space with a metric of the form  $(u, v) = \int (\text{grad } u \cdot \text{grad } v) dg$ . [See Weyl, *Duke Math. J.* 7 (1940), 411-444; MR 2, 202.]

The alternating process corresponds to a sequence of operators:  $P_1, P_2 P_1, P_1 P_2 P_1, \dots$  which, according to a theorem of Wiener [*Comment. Math. Helv.* 29 (1955), 97-111; MR 16, 921] converges to a limiting operator with the desired properties.  
J. W. Green.

**Walsh, J. L.** On the conformal mapping of multiply connected regions. *Trans. Amer. Math. Soc.* **82** (1956), 128-146.

The basic result of this paper is as follows: Let  $D$  be a region of the extended  $z$ -plane whose boundary consists of mutually disjoint Jordan curves  $B_1, B_2, \dots, B_n$ ;  $C_1, C_2, \dots, C_\nu$ ,  $\mu \neq \nu$ . There exists a conformal map of  $D$  onto a region  $\Delta$  of the extended  $Z$ -plane, one to one and continuous in the closures of the two regions, where  $\Delta$  is defined by

$$1 < |T(Z)| < \exp(1/\tau), \quad T(Z) = A \prod_{j=1}^{\mu} (Z - a_j)^{M_j} \prod_{k=1}^{\nu} (Z - b_k)^{-N_k}, \\ \sum_{j=1}^{\mu} M_j = \sum_{k=1}^{\nu} N_k = 1, \quad \tau > 0.$$

The exponents  $M_j$  and  $N_k$  are positive but need not be rational. The locus  $|T(Z)|=1$  consists of  $\mu$  mutually disjoint Jordan curves, respective images of the  $B_j$ , which separate  $\Delta$  from the  $a_j$ ; the locus  $|T(Z)|=\exp(1/\tau)$  consists of  $\nu$  mutually disjoint Jordan curves, respective images of the  $C_k$  which separate  $\Delta$  from the  $b_k$ .

This result may be regarded as the culmination of results of de la Vallée Poussin and Julia on canonical conformal mappings associated with level curves of polynomials and rational functions. The author allows the transcendental functions in the above statement replacing the rational functions and thus is able to obtain a completeness of representation lacking in the earlier works. The principal feature of the author's method consists of approximation to the harmonic measure of the union of the curves  $C_j$  with respect to  $D$  by the potential of a finite set of discrete positive and negative masses. In this way  $D$  is approximated by a sequence of domains  $D_n$  bounded by level curves of rational functions. The desired mapping is obtained by a passage to the limit executed on similar mappings for the domains  $D_n$ .

Corresponding to a given division of the boundary components of  $D$  into two classes the canonical mapping for  $D$  is unique up to a linear transformation of the  $Z$ -sphere.

The author obtains an analogous result for the case in which each member of the first set of boundary components is allowed to degenerate to a point. A uniqueness result of the same type as before is obtained.

This paper is an elaboration and extension of two earlier notes of the author [*C. R. Acad. Sci. Paris* **239** (1954), 1572-1574, 1756-1758; *MR* **16**, 581, 811].

{It is of interest to note that the results of this paper are readily obtained by a suitable development of considerations given in a paper by the reviewer and D. C. Spencer [*Ann. of Math.* (2) **53** (1951), 4-35; *MR* **12**, 400]. The reviewer hopes to give complete details of this elsewhere.}

J. A. Jenkins (Notre Dame, Ind.).

**Položii, G. N.** On an addendum to a theorem on motion of boundary points. *Ukrain. Mat. Ž.* **7** (1955), 339-342. (Russian)

The author has shown [*Uspehi Mat. Nauk* (N.S.) **7** (1952), no. 6(52), 203-205; *MR* **14**, 549] that if  $G$  is a simply-connected region lying inside another simply-connected region  $G'$  in such a way that  $G$  and  $G'$  have a Jordan arc  $\gamma$  as a common part of their frontiers, then, under a conformal mapping of  $G$  onto  $G'$ , there can be no more than three fixed points in the interior of  $\gamma$ . In the event that there are three fixed points, the points of  $\gamma$  gravitate towards the outer two under the conformal map and are repelled from the middle fixed point, while, in the

case of two fixed points, one of the points attracts and the other repels. In the present note, the author shows that, if the derivative of the mapping function exists on  $\gamma$ , its modulus is less than unity at an attractive fixed point and greater than unity at the repellent fixed point.

A. J. Lohwater (Ann Arbor, Mich.).

**Mori, Shin'ichi; and Ota, Minoru.** A remark on the ideal boundary of a Riemann surface. *Proc. Japan Acad.* **32** (1956), 409-411.

The authors show that the set  $\Delta$  of ideal boundary points of an open Riemann surface [cf. Royden, *Contributions to the theory of Riemann surfaces*, Princeton, 1953, pp. 107-109; *MR* **15**, 25] has the property that every function in the ring of harmonic functions with a finite Dirichlet integral assumes its maximum on  $\Delta$ . The proof makes use of Gelfand's theory of normed rings.

H. L. Royden (Stanford, Calif.).

**Oikawa, Kôtarô.** Notes on conformal mappings of a Riemann surface onto itself. *Kôdai Math. Sem. Rep.* **8** (1956), 23-30.

The author is concerned with determining bounds for the number of conformal automorphisms of a Riemann surface of finite genus with a finite number of boundary elements onto itself. Let  $N(g, k)$  denote the supremum of the number of conformal automorphisms of a Riemann surface of finite genus  $g$  whose boundary consists of  $k$  ( $\geq 1$ ) contours, all such surfaces being taken into account. Similarly let  $N'(g, k)$  have a corresponding meaning relative to the class of compact Riemann surfaces of genus  $g$  punctured in  $k$  points. It is shown that: if  $2g+k \geq 3$ , then  $N'(g, k) \leq N(g, k) \leq 12(g-1)+6k$ . The equality of  $N'(g, k)$  and  $N(g, k)$  is announced. The proof makes use of doublings and the Riemann-Hurwitz relation. For the case  $g=1$ ,  $N'(1, k)$  is determined completely in terms of  $k$ . The value of  $N'(1, k)$  depends on number-theoretic properties of  $k$ .

M. H. Heins (Princeton, N.J.).

**Stollow, S.** Sur la classification topologique des recouvrements riemanniens. *Rev. Math. Pures Appl.* **1** (1956), no. 2, 37-42.

Riemannian coverings are classified according to their boundary behavior. Applications are given to certain problems of classical analysis (differential equations, implicit functions, etc.).

M. H. Heins.

**Kuramochi, Zenjiro.** Evans-Selberg's theorem on abstract Riemann surfaces with positive boundaries. I, II. *Proc. Japan Acad.* **32** (1956), 228-233, 234-236.

The author defines an ideal boundary  $B$  of the Martin type for an open Riemann surface  $R$  by using the Neumann's function of the surface. He then proceeds to construct equilibrium mass distributions on certain closed subsets  $R \cup B$ . H. L. Royden (Stanford, Calif.).

**Kuroda, Tadashi.** On analytic functions on some Riemann surfaces. *Nagoya Math. J.* **10** (1956), 27-50.

A theorem of Iversen states that if  $w=f(z)$  is meromorphic in the complex plane, then any function element of the inverse of  $f$  at a point  $w_0$  can be continued analytically up to any other point  $w$  by an analytic continuation interior to any disc centered at  $w$  containing  $w_0$ .

The author extends this theorem to the case of functions meromorphic on a Riemann surface of class  $O_{AB}$  but not  $O_{HD}$ .

H. L. Royden (Stanford, Calif.).

Sunyer Balaguer, F. Directions of Borel-Valiron of maximum kind common to an entire function and to its successive derivatives and integrals. Mem. Acad. Ci. Madrid 5 (1956), no. 1, 51 pp. (Spanish)

Milloux [J. Analyse Math. 1 (1951), 244-330; MR 13, 930] showed that for an entire function of positive order there is at least one Borel direction which the function has in common with all its derivatives and integrals. The author sharpens this by showing the same for Borel directions of the maximum kind, and gives some further results of similar character. He then shows that the number of Borel directions of the maximum kind is connected with the number of exceptional values, for example as follows: if  $f(z)$  is of order at least  $3/2$ , of type  $b$  of the proximate order  $\rho(r)$ , and if there are fewer than  $2m$  directions of Borel of maximum kind common to  $f$  and all its derivatives and integrals, where  $m$  is the integer closest to  $\rho$ ; then for all finite  $a$  and all (positive or negative) integers  $k$ ,

$$\limsup_{r \rightarrow \infty} r^{-\rho(r)} n(r, 1/(f^{(k)} - a)) \geq bB(\rho),$$

where  $B$  depends only on  $\rho$ . Finally the author shows that for a function with a lacunary power series whose exponents have upper density  $D$  one can infer the existence not only of Julia lines in every angle of opening exceeding  $2\pi \max(D, \frac{1}{2}\rho^{-1})$ , but even of Borel directions of maximum kind for all derivatives and integrals.

R. P. Boas, Jr. (Evanston, Ill.).

Krein, M. G. Continuous analogues of propositions on polynomials orthogonal on the unit circle. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 637-640. (Russian)

Let  $P_n(z)$  ( $n=0, 1, 2, \dots$ ) be a set of polynomials orthogonal on the unit circle. The author gives a concentrated summary of an extensive parallel theory of certain families of entire functions  $P(r; \lambda)$  of  $\lambda$ , where  $r$  (corresponding to  $n$ ) may have any value  $\geq 0$ , and the real  $\lambda$ -axis takes the place of the unit circle. The results given correspond to the deeper analytic properties of the  $P_n(z)$ . The notation is chosen so as to be analogous to that used by Ya. L. Geronimus [see, e.g., Izv. Akad. Nauk SSSR. Ser. Mat. 14 (1950), 123-144; MR 12, 177; 15, 217].

The  $P_n(z)$  are derivable from either (1) a Toeplitz matrix  $\{h_{n-m}\}$ , (2) a weight-distribution  $d\sigma(\theta)$ , or (3) the recurrence relation  $P_{n+1}(z) = zP_n(z) - a_n P_n^*(z)$ , where  $P_n^*(z) = z^n \bar{P}_n(1/\bar{z})$ . The author adopts the analogue of (1), (2) and (3) being then consequential. He considers a Hermitean kernel  $H(t-s)$  such that, for every continuous  $\phi(t)$ ,

$$\int_0^r |\phi(s)|^2 ds + \int_0^r \int_0^r H(t-s) \phi(t) \bar{\phi}(s) dt ds \geq 0 \quad (0 < r < \infty),$$

with equality only if  $\phi(s) \equiv 0$ ; a spectral function  $\sigma(\lambda)$ , in terms of which  $H(t-s)$  can be expressed (cf. (2)), then exists. Defining the resolvent  $\Gamma_r(t, s)$  by

$$\Gamma_r(t, s) + \int_0^r H(t-u) \Gamma_r(u, s) du = H(t-s) \quad (0 \leq s, t \leq r),$$

he puts (cf. the determinantal expression for  $P_n(z)$ )

$$P(r; \lambda) = e^{i\lambda r} (1 - \int_0^r \Gamma_r(s, 0) e^{-i\lambda s} ds),$$

$$P_*(r; \lambda) = 1 - \int_0^r \Gamma_r(0, s) e^{i\lambda s} ds.$$

Then  $P(r; \lambda)$ ,  $P_*(r; \lambda)$  satisfy differential equations in  $r$  (cf. (3)), and  $P(r; \lambda)$  has no zeros in  $\text{Im } \lambda < 0$ . Properties of the mapping  $F(\lambda) = \int_0^\infty f(r) P(r; \lambda) dr$  are announced.

Next given are four properties of the  $P(r; \lambda)$ ,  $P_*(r; \lambda)$  which are separately equivalent to the finiteness of  $\int_0^\infty \ln \sigma'(\lambda)/(1+\lambda^2) d\lambda$ ; these include the finiteness for at least one  $\lambda$  with  $\text{Im } \lambda > 0$  of  $\int_0^\infty |P(r; \lambda)|^2 dr$  and the existence for  $\text{Im } \lambda > 0$ , uniformly in a bounded closed region, of the limit  $\Pi(\lambda) = \lim_{r \rightarrow \infty} P_*(r; \lambda)$ . A number of interesting properties of  $\Pi(\lambda)$  are stated.

The rest of the paper announces properties of the functions  $\mathcal{E}(r; \lambda) = e^{-i\lambda r} P(2r; \lambda) = \Phi(r; \lambda) + i\Psi(r; \lambda)$ , of the mapping  $\mathcal{F}(\lambda) = \int_0^\infty f(r) \mathcal{E}(r; \lambda) dr$ , and of the differential equations satisfied by  $\Phi$  and  $\Psi$ . The investigation provides a solution of the inverse spectral problem for the latter differential equations; alternatively, these differential equations can be a starting point for the theory.

F. V. Atkinson (Canberra).

Parreau, Michel. Fonction caractéristique d'une application conforme. Relation avec la notion d'application de type Bl. C. R. Acad. Sci. Paris 241 (1955), 1545-1546.

L'auteur annonce de nombreux résultats concernant une généralisation de la fonction caractéristique  $T(r, f)$  de Nevanlinna, pour une application d'un domaine Greenien sur un autre. A l'origine de la définition de cette fonction caractéristique est une extension de l'inégalité de Lehto, et ses propriétés sont à rapprocher de celles de la fonction caractéristique généralisée par M. Heins [Ann. of Math. (2) 62 (1955), 418-446, p. 424, encore sous presse au moment de la publication de la présente note; MR 17, 726]. L'auteur signale le lieu avec les applications de type Bl définies et étudiées par M. Heins [ibid. 61 (1955), 440-473; MR 16, 1011] et les rapports avec le théorème de Fatou sur les limites radiales (ce point est également étudié par Heins dans "Lindelöfian maps") étendu ici aux domaines Greeniens.

L. Fourès (Marseille).

Parreau, Michel. Fonction caractéristique d'une application conforme. Ann. Fac. Sci. Univ. Toulouse (4) 19 (1955), 175-190 (1956).

Le présent mémoire est le développement, contenant toutes les démonstrations, de la note analysée ci-dessus; il est consacré à une extension de la notion de fonction caractéristique  $T(r, f)$  aux applications d'un domaine Greenien sur un autre. Extension des travaux de Lehto et M. Heins. Aux résultats déjà indiqués dans la note aux C. R. citée on peut ajouter un théorème intéressant concernant les applications d'une surface Greenienne sur une surface non Greenienne: le résultat sur les limites suivant les lignes de Green s'étend alors à la "frontière idéale" de la surface image. L. Fourès (Marseille).

Ohtsuka, Makoto. Generalizations of Montel-Lindelöf's theorem on asymptotic values. Nagoya Math. J. 10 (1956), 129-163.

Given the strip  $B: 0 < x < \infty, 0 < y < 1$ ; in the  $z$ -plane, Montel [Ann. Sci. Ecole Norm. Sup. (3) 29 (1912), 487-535] proved that a function  $f(z)$  analytic in  $B$  and continuous in  $0 < x < \infty, 0 \leq y < 1$ , which omits at least two values in  $B$ , necessarily tends to  $w_0$  as  $z \rightarrow \infty$  in  $0 \leq y < 1 - \varepsilon$  ( $\varepsilon > 0$ ), if it does so on the real axis. The convergence to  $w_0$  in  $\varepsilon < y < 1 - \varepsilon$  was proved by Lindelöf [Acta Soc. Sci. Fenn. 46 (1915), no. 4] for bounded analytic functions in  $B$  tending to  $w_0$  along a curve in  $B$  which tends to  $z = \infty$  and this result was generalized by Gross [Math. Z. 3 (1919), 43-64] to exceptionally ramified (ausnahmsverzweigt)

closed set  $F$  on the real axis is said to have positive average logarithmic capacity (or positive average linear functions. The author gives further generalizations. A measure) near  $x=\infty$ , if there exist numbers  $x_0$ ,  $a>0$  and  $d>0$  such that the logarithmic capacity (or the linear measure) of the part of  $F$  in the interval  $(x-a, x+a)$  is greater than  $d$  for all  $x>x_0$ . The definition of an  $(\mathfrak{L})$  parabolic transformation of schlicht type of  $B$  into a connected topological space  $\mathfrak{F}^*$  is given, where  $\mathfrak{L}$  is a new element defined by means of a filter in  $\mathfrak{F}^*$ . If  $F$  is a closed set on the real axis of positive average logarithmic capacity near  $x=\infty$ ,  $f(z)$  an  $(\mathfrak{L})$  parabolic transformation of schlicht type of  $B$  into  $\mathfrak{F}^*$  and a continuous transformation of  $B+F$  into  $\mathfrak{F}^*+\{\mathfrak{L}\}$  for which  $f(x)\rightarrow\mathfrak{L}$  as  $x\rightarrow\infty$  in  $F$ , we have  $f(z)\rightarrow\mathfrak{L}$  as  $z\rightarrow\infty$  in  $B$  outside a relatively closed set  $\Omega$  in  $B$  which tends to the boundary of  $B$  for  $z\rightarrow\infty$  and such that the extremal distance of  $\Omega$  and  $F$  within the part  $x>x_1$  of  $B$  tends to  $\infty$  as  $x_1\rightarrow\infty$ . The theorem is applied to conformal mapping and generalized to  $(\mathfrak{L})$  parabolic transformations, a class of pseudo-analytic functions defined by the author in Nagoya Math. J. 9 (1955), 191-207 [MR 17, 1191]. An analytic transformation of  $B$  into a Riemann surface  $\mathfrak{R}$  is called exceptionally ramified in the generalized sense if either (1) the genus of  $\mathfrak{R}$  is  $>2$  or (2), in case of a planar  $\mathfrak{R}$ , there exist points  $P_k$  in  $\mathfrak{R}$  and integers  $\mu_k\geq 2$ ,  $\sum_k (1-\mu_k)>2$ , such that all except finitely many points of the Riemann surface of the inverse function situated above  $P_k$  are branch-points of order divisible by  $\mu_k$  (Gross) or (3), if  $\mathfrak{R}$  is a torus, there exists at least one such point  $P_0$  and  $\mu_0\geq 2$ ; if  $P_k$  is not covered at all, we set  $\mu_k=\infty$ . A new element  $\mathfrak{L}$  of a Riemann surface  $\mathfrak{R}$  defined by means of a filter in  $\mathfrak{R}$  is said to be (complete and) of harmonic measure zero, if there exists a function  $v(P)$ , superharmonic outside a certain point  $P_0$ , bounded from below outside every neighbourhood of  $P_0$  and tending to  $\infty$  if (and only if)  $P\rightarrow\mathfrak{L}$ . If  $F$  is a closed set on the  $x$ -axis of positive average linear measure near  $x=\infty$ ,  $\mathfrak{L}$  a complete element of harmonic measure zero added to the Riemann surface  $\mathfrak{R}$  with positive boundary,  $f(z)$  an exceptionally ramified analytic transformation of  $B$  into  $\mathfrak{R}$  and a continuous transformation of  $B+F$  into  $\mathfrak{R}+\{\mathfrak{L}\}$  tending to  $\mathfrak{L}$  as  $x\rightarrow\infty$  on  $F$ , we conclude  $f(z)\rightarrow\mathfrak{L}$  in any  $\varepsilon<y<1-\varepsilon$ ,  $\varepsilon>0$ . A counterexample is provided for the case that  $F$  has no positive average linear measure near  $x=\infty$ . If  $\mathfrak{R}$  has null-boundary, the convergence in  $0\leq y<1-\varepsilon$  is concluded from the convergence on the whole  $x$ -axis and an example of a function converging outside a closed set of linear measure zero on the  $x$ -axis, which has no definite limit as  $z\rightarrow\infty$  along any curve in  $B$ , is given. *K. Strebel.*

**Kawakami, Yoshiro.** On Montel's theorem. Nagoya Math. J. 10 (1956), 125-127.

The author proves the following generalization of Montel's theorem [see the preceding review], using Green's function of a half disk: Let  $E$  be a measurable set on the positive  $y$ -axis with positive lower density  $\lambda=\liminf_{r\rightarrow 0}\mu(r)\cdot r^{-1}$  at  $y=0$ , where  $\mu(r)$  denotes the linear measure of the part of  $E$  in the interval  $0\leq y\leq r$ . A bounded analytic function  $f(z)$  in  $x>0$  with continuous boundary at  $E$  which tend to  $A$  for  $y\rightarrow 0$ , necessarily tends to  $A$  in any angle  $|y|\leq k\cdot x$ ,  $k>0$ . The same is true if

$$\lambda_\alpha=\liminf_{r\rightarrow 0} r^{\alpha-1} \int_r^1 \frac{d\mu(t)}{t^\alpha} \quad (\alpha\geq 2)$$

is positive.

*K. Strebel (Fribourg).*

**Ohtsuka, Makoto.** Remarks to the paper "on Montel's theorem" by Kawakami. Nagoya Math. J. 10 (1956), 165-169.

The author proves the equivalence of Kawakami's conditions [see the preceding review]  $\lambda>0$  and  $\lambda_\alpha>0$  for any  $\alpha>1$ , and also the equivalence of the two conditions that a closed set  $F$  on the real axis be of positive average linear measure near  $x=\infty$  [see the second preceding review] and its image of positive lower density at  $y=0$  if the strip  $B: 0<x<\infty$ ,  $0<y<1$  is mapped conformally onto the half plane  $\xi>0$  ( $\zeta=\xi+i\eta$ ) with  $z=\infty\leftrightarrow\zeta=0$ .

*K. Strebel (Fribourg).*

**Ohtsuka, Makoto.** On boundary values of an analytic transformation of a circle into a Riemann surface.

Nagoya Math. J. 10 (1956), 171-175.

The author proves by means of methods developed in the paper reviewed above the following theorem. If  $f(z)$  is a nonconstant exceptionally ramified analytic transformation of  $|z|<1$  into a Riemann surface  $\mathfrak{R}$ ,  $\mathfrak{L}$  an element of harmonic measure zero, and  $E$  a set on  $|z|=1$  at each point of which there terminates a curve along which  $f(z)$  tends to  $\mathfrak{L}$ , the linear measure of  $E$  is zero. *K. Strebel.*

**Komatu, Yūsaku.** A coefficient problem for functions univalent in an annulus. Kōdai Math. Sem. Rep. 8 (1956), 49-70.

The author studies the family  $\mathfrak{F}_R$  of functions  $w=f(z)=\sum_{n=-\infty}^{\infty} c_n z^n$  which have the following properties: (i)  $f(z)$  is regular, univalent and of absolute value greater than 1 in the annulus  $1<|z|<R$ ; (ii) the circumference  $|z|=1$  is mapped onto  $|w|=1$  and  $f(1)=1$ . Grötzsch's extremal function  $f_R(z)=\sum c_n^* z^n$  which maps the annulus onto the exterior of the unit circle cut along a half-line on the negative real axis is used as comparison function. The author expresses  $f_R(z)$  in terms of elliptic functions and computes the coefficients  $c_n^*=E n R^{-n}/(1-R^{-4n})$ ,  $n\neq 0$ , where  $E$  is a constant. No function of  $\mathfrak{F}_R$  can be a universal extremal function for the coefficient problem for  $\mathfrak{F}_R$ , but the author conjectures that at least

$$\limsup |c_n|/c_n^* \leq 1$$

as  $n\rightarrow+\infty$  for every  $f(z)$  of  $\mathfrak{F}_R$ . He proves that

$$|c_n| < e c_n^* + O(R^{-n})$$

with an  $O$  which is uniform with respect to  $n$  as well as  $f(z)$ . Various other inequalities involving coefficients are proved. Some of these lead to estimates for  $c_n$  as  $n\rightarrow-\infty$ .

*J. Korevaar (Madison, Wis.).*

**Alenicyan, Yu. E.** On univalent functions in multiply connected domains. Mat. Sb. N.S. 39(81) (1956), 315-336. (Russian)

The paper contains the proofs of results announced in a previous paper (P) [Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 861-863; MR 17, 25], with some extensions and additions. Results not found in (P) are as follows.

Let  $f(z)$  map the region  $B$  conformally onto the bounded region  $D$ , with  $L$  mapping onto the outer boundary component of  $D$ . For each  $z_1\in B$ , the precise bound  $|f'(z_1)|\leq R|f'(z_1, L)|$  holds, where  $R$  is the outer conformal radius of the exterior of the outer boundary component of  $D$ , and  $f(z, L)$  is a certain function depending on  $B$  and  $L$ , but not on  $D$ .

Let  $K$  be the interior of the unit circle, less a finite number of concentric circular slits, and  $S_1(K)$  the class of functions  $f(z)$ ,  $f(0)=0$ ,  $|f(z)|<1$ , regular and schlicht in  $K$ ,

such that  $|z|=1$  corresponds to the exterior boundary component  $C$  of the image. Then, if  $c$  is a point of  $\bar{C}$ ,  $|f'(0)| \leq 4|c|/(1+|c|)^2$ . E. Reich (Minneapolis, Minn.).

**Alenicyan, Yu. E.** A contribution to the theory of univalent and Bieberbach-Eilenberg functions. Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 247-249. (Russian)

Let  $f(z)$ ,  $f(0)=0$ , be regular and schlicht, and  $F(z)$ ,  $F(0)=\infty$ , be meromorphic and schlicht in  $U: |z| < 1$ . If the image of  $U$  under  $f$  is disjoint from that under  $F$ , then for any  $z, z' \in U$ ,

$$(*) \quad \left| \log \left( 1 - \frac{f(z)}{F(z')} \right) \right| \leq -\frac{1}{2} \log[(1-|z|^2)(1-|z'|^2)].$$

The derivation is based on an inequality due to Nehari [Trans. Amer. Math. Soc. 75 (1953), 256-286; MR 15, 115].

Using (\*), some best inequalities for the class  $C$  of Bieberbach-Eilenberg functions  $f(z)$ ,  $f(0)=0$ , regular in  $U$ , and such that  $f(z_1)/f(z_2) \neq 1$  for  $z_1, z_2 \in U$ , are derived. For instance,  $f \in C$  implies  $1-|z|^2 \leq |1-f^2(z)| \leq (1-|z|^2)^{-1}$ . [The right side of this inequality is an immediate consequence of a theorem of Jenkins, *ibid.* 76 (1954), 389-396; MR 16, 24.] Other, more complicated, inequalities involve the subclass  $C_*$  of schlicht Bieberbach-Eilenberg functions.

E. Reich (Minneapolis, Minn.).

**Kac, I. S.** On integral representations of analytic functions mapping the upper half-plane onto a part of itself. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 3(69), 139-144. (Russian)

A number of integral representations for the class  $(R)$  of functions  $f(z)$ , regular for  $\text{Im} z \neq 0$ ,  $f(\bar{z}) = \overline{f(z)}$ ,  $\text{Im} f(z)/\text{Im} z > 0$ , are considered. Theorem 2: A necessary and sufficient condition for  $f \in (R)$  to have the absolutely convergent representation  $f(z) = c + \int_{-\infty}^{\infty} d\tau(t)/(t-z)$ ,  $\text{Im} c = 0$ ,  $\tau(t)$  non-decreasing, is that  $\int_{-\infty}^{\infty} \text{Im} f(i\eta)/\eta \, d\eta < \infty$ . Another representation studied is

$$f(z) = a + \int_{-\infty}^{\infty} \left( -\frac{i}{1+t^2} + \frac{1}{t-z} \right) d\tau(t), \quad \text{Im} a = 0,$$

the possibility of which, for a certain class of  $\tau(t)$ , depends on the convergence or divergence of  $\int_{-\infty}^{\infty} \text{Im} f(i\eta)/\eta^\alpha \, d\eta$  for some  $\alpha$ ,  $0 < \alpha < 2$ . E. Reich (Minneapolis, Minn.).

**Flett, T. M.** On some theorems of Littlewood and Paley. J. London Math. Soc. 31 (1956), 336-344.

Let  $\phi(z)$  be regular for  $|z| < 1$ , and let

$$g^*(\theta, \phi) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^1 (1-\rho) |\phi'(\rho e^{it})|^2 P(\rho, t-\theta) \, d\rho dt, \right.$$

where  $P(\rho, t)$  is the Poisson kernel. The function  $g^*$  was introduced by Littlewood and Paley [Proc. London Math. Soc. (2), 42 (1936), 52-89]. The author deals with the following inequality, due to Zygmund [Trans. Amer. Math. Soc. 55 (1944), 170-204; MR 5, 230]

$$(*) \quad \left( \int_{-\pi}^{\pi} (g^*(\theta, \lambda d\theta))^{1/\lambda} \, d\theta \leq A(\lambda) \left( \int_{-\pi}^{\pi} |\phi(e^{it})|^{1/\lambda} \, d\theta \right)^{1/\lambda} \quad (1 < \lambda < \infty),$$

and related inequalities for the function

$$g(\theta, \phi) = \left( \int_0^1 (1-\rho) |\phi'(\rho e^{it})|^2 \, d\rho \right)^{1/2},$$

and the "area integral" of Lusin. The author simplifies the chain of deductions which culminate in (\*), by

judicious use of the arguments previously employed by Littlewood and Paley, and Zygmund.

The author also obtains the following partial converse of (\*), when the function  $g$  is substituted for the function  $g^*$ . It is assumed that  $\psi(z)$  has no zeroes within the unit circle, then:

$$\int_{-\pi}^{\pi} |\psi(e^{it\theta})|^\mu \, d\theta \leq A(\mu) \int_{-\pi}^{\pi} g^\mu(\theta, \psi) \, d\theta + A(\mu) |\psi(0)|^\mu \quad (0 < \mu \leq 1).$$

For  $1 < \mu$ , an analogue of the above without restriction on the zeroes of  $\psi(z)$  had previously been obtained by Littlewood and Paley (*loc. cit.*). E. M. Stein.

**Nehari, Zeev.** On the singularities of Legendre expansions. J. Rational Mech. Anal. 5 (1956), 987-992.

Szegő [same J. 3 (1954), 561-564; MR 16, 34] proved that if  $\{a_n\}$  is a real sequence such that  $\limsup |a_n|^{1/n} = 1$  and  $u(r, \theta)$  denotes the harmonic function given by  $\sum_{n=0}^{\infty} a_n r^n P_n(\cos \theta)$ ,  $0 \leq \theta \leq 2\pi$ ,  $0 \leq r < 1$ , then  $(1, \theta)$  is a regular point of  $u$  if and only if  $e^{i\theta}$  is a regular point of  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ . The following generalization of his result is established: Let  $\limsup |a_n|^{1/n} = \zeta < 1$ , and let  $g(t)$  and  $f(t)$  be analytic functions defined by  $\sum_{n=0}^{\infty} a_n P_n(t)$ ,  $|t+1| + |t-1| < (1+\zeta^2)\zeta^{-1}$ , and  $\sum_{n=0}^{\infty} a_n z^n$ ,  $|z| < \zeta^{-1}$ , respectively. Then a point  $t=\tau$  ( $\tau \neq \pm 1$ ), which is reached by analytic continuation along a path  $C_t$  originating at  $t=1$  and otherwise avoiding the points  $t=\pm 1$ , will be a singularity of  $g$  if, and only if,  $\tau = (1+\sigma^2)/(2\sigma)$ , where  $z=\sigma$  is a singularity of  $f$  which is reached by analytic continuation of  $f$  along a path  $C_z$  originating at  $z=1$  and coinciding with one of the two images of  $C_t$  under the conformal mapping  $t=(1+z^2)/(2z)$ . Szegő's result corresponds to the limiting case  $\sigma \rightarrow e^{i\theta}$ . If the  $a_n$  are real, the singularities of  $f$  appear in conjugate pairs. This accounts for Szegő's result and the fact that it will not hold unless the  $a_n$  are real.

N. D. Kazarinoff (Ann Arbor, Mich.).

**MacLane, Gerald R.** Limits of rational functions. Pacific J. Math. 6 (1956), 111-116.

Laguerre, Pólya and others, among them the reviewer [Duke Math. J. 21 (1954), 533-548; 18 (1951), 573-592; MR 16, 347; 13, 222] have studied convergence and limit functions of sequences of polynomials whose zeros are restricted to a suitable portion of the plane. Uniform convergence in a suitable domain, to a non-zero limit, implies uniform convergence in every bounded domain, and the limit functions are entire functions of very special form. Comparable results for sequences of rational functions are known only if poles and zeros are kept away from each other [cf. Obrechhoff, Quelques classes de fonctions entières ..., Hermann, Paris, 1941; MR 7, 516]. The author gives examples which show that if zeros and poles are allowed to mix freely there need be no convergence outside the original domain of convergence while any analytic function which satisfies some obvious conditions can occur as limit function. In these examples zeros and poles are real and the domains of convergence are symmetric with respect to the real axis.

J. Korevaar (Madison, Wis.).

**Hayman, W. K.** A generalisation of Stirling's formula. J. Reine Angew. Math. 196 (1956), 67-95.

The author's main result is an asymptotic formula of remarkable generality for the coefficients of a wide class of analytic functions. Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ , regular in

$|z| < R \leq \infty$ , real for real  $z$ . Put  $a(r) = r f'(r)/f(r)$ ,  $b(r) = r a'(r)$ , suppose  $b(r) \rightarrow \infty$ , and suppose further that with some  $\delta(r)$ ,  $0 < \delta(r) < \pi$ , we have  $f(re^{i\theta}) \sim f(r)e^{i\theta a(r) - i\theta^2 b(r)}$  uniformly for  $|\theta| \leq \delta(r)$ , while  $f(re^{i\theta}) = o(f(r))/b(r)^k$  uniformly for  $\delta(r) \leq |\theta| \leq \pi$ . Such an  $f$  is called admissible. Then, uniformly in  $n$ ,

$$a_n r^n = \frac{f(r)}{\{2\pi b(r)\}^k} \left\{ \exp \left[ -\frac{(a(r)-n)^2}{2b(r)} \right] + o(1) \right\};$$

there are a number of corollaries about the behaviour of  $a_n$ ,  $a(r)$ ,  $b(r)$ , and of the maximum term and its index. After proving this the author takes up the question of what functions are admissible. He shows that for admissible functions the distribution of the partial sums of  $\sum a_n r^n$  is asymptotically Gaussian, that  $f^{(k)}(r) \sim f(r)[a(r)]^k$ , that  $f_k(r) = \inf_{h>0} h^{-k} f(r+h) \sim f(r)[ea(r)/(kr)]^k$ , and that for sufficiently large  $r < R$  and  $0 < |\theta| \leq \pi$  we have  $|f(re^{i\theta})| < f(r)$ , so that  $M(r) = f(r)$  for  $r$  sufficiently near  $R$ . If  $f$  and  $g$  are admissible, so are  $ef$  and  $fg$ . Let  $P(z) = b_0 + b_1 z + \dots + b_m z^m$  be a real polynomial with  $P(R) > 0$  if  $R$  is finite,  $b_m > 0$  if  $R = \infty$ . Then  $fP$  is admissible, and so (without restriction on the  $b$ 's) is  $f+P$  (or indeed  $f+g$  where  $g$  is essentially smaller than  $f$ ) and also  $P[f(z)]$  if  $b_m > 0$ . By using these facts one can build up many admissible functions out of the following examples. First  $e^{P(z)}$  is admissible if and only if it attains its maximum modulus on the positive real axis, several equivalent conditions being given. Next, an entire function of genus 0 is admissible if it is positive for large positive  $z$ , has, for some positive  $\delta$ , at most a finite number of zeros in  $|\arg z| \leq \frac{1}{2}\pi + \delta$ , and is such that  $b(r) \rightarrow \infty$ . An example is  $\Xi(z^4)$ . For functions of genus 1 this is no longer true, and the author illustrates this point by a detailed discussion of  $1/\Gamma(z)$ : here there is a sequence of integers  $n_p \rightarrow \infty$  such that  $n_{p+1} - n_p \sim \log n_p$  and  $a_{n_p} a_{n_p+1} > 0$  while  $(-1)^n a_n$  has a constant sign for  $n_p + 1 \leq n \leq n_{p+1}$ . For functions in the unit disk,  $f$  is admissible if positive on the real axis near 1, if further there are positive constants  $\alpha, \beta$  ( $\beta < 1$ ) and a positive  $C(r)$  such that  $(1-r)C'(r)/C(r) \rightarrow 0$  and  $\log f(z) \sim C(|z|)(1-z)^{-\alpha}$  uniformly for  $|\arg z| \leq \beta(1-r)$ ; and if finally  $|f(re^{i\theta})| \leq |f(re^{i\beta(1-r)})|$ ,  $\beta(1-r) \leq |\theta| \leq \pi$ , for  $r$  near 1. In particular if  $f(z) = \sum a_n z^n$  is regular in  $|z| < 1$  and  $|f(z)| > 1$ , and  $F(z) = |a_0|^{(1+z)/(1-z)} = |a_0| + \sum_{n=1}^{\infty} A_n z^n$ , then  $|a_n| \leq A_n$  and the asymptotic value of the  $A_n$  [Macintyre and Wilson, Math. Ann. 127 (1954), 243-250; MR 15, 946] is obtained.

R. P. Boas, Jr.

**Trošín, G. D.** On the interpolation of functions analytic in an angle. Mat. Sb. N.S. 39(81) (1956), 239-252. (Russian)

The author studies the interpolation problem in the class  $W_\rho$  of functions  $f(z)$ , which are analytic in some angle  $|\arg z| < \pi/2\rho$  ( $\frac{1}{2} \leq \rho < \infty$ ) and which have a limit as  $z \rightarrow 0$  in the angle and a growth condition of the type

$$|f(z)| < \exp[|z|^\rho + \varepsilon] \quad (\varepsilon > 0)$$

for large  $|z|$ . It is shown that, in order to find at least one function  $f(z)$  in  $W_\rho$  such that  $f(\lambda_n) = a_n$  ( $n=1, 2, \dots$ ),  $0 < |\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_n| \leq \dots$ , subject to the restriction

$$\limsup_{n \rightarrow \infty} \frac{\log \log |a_n|}{\log |\lambda_n|} \leq \rho$$

it is necessary and sufficient that the sequence  $\lambda_n$  satisfy the conditions (i)  $\sum |\lambda_n|^{-\rho-\varepsilon} < \infty$ , and

$$(ii) \quad \limsup_{n \rightarrow \infty} \frac{1}{\log |\lambda_n|} \log \log \frac{1}{\varphi'(\lambda_n)} \leq \rho,$$

where

$$\varphi(z) = \prod_{n=1}^{\infty} \frac{1 - z^m (\lambda_n^m)^{-1}}{1 + z^m (\lambda_n^m)^{-1}}.$$

A. J. Lohwater (Ann Arbor, Mich.).

**deLeeuw, K.** A type of convexity in the space of  $n$ -complex variables. Trans. Amer. Math. Soc. 83 (1956), 193-204.

Let  $H$  be an abelian group generated by  $s_1, \dots, s_n$ , and  $SCH$  be the semi-group of elements  $\{s_1^{m_1} \dots s_n^{m_n}\}$ , where  $m_i$  are non-negative integers, not all of which are zero. Let  $V$  be the point-set union of the images of all maps  $\varphi: S \rightarrow C^n$  for which  $\varphi(s \cdot s') = \varphi(s)\varphi(s')$ ; here  $C^n$  is the space of  $n$  complex variables, and multiplication in  $C^n$  is taken componentwise. A subset  $WCV$  is called  $V$ -circular if, whenever  $v = (v_i) \in V$ ,  $w = (w_i) \in W$ , and  $|v_i| = |w_i|$  for all  $i$ , then  $v \in W$ ; in case  $V = C^n$ , this is just the condition that  $W$  be a Reinhardt domain with center at the origin. The principal result of the paper is an alternative characterization of the polynomial completions of  $V$ -circular domains. More precisely, for any compact  $XCC^n$  the polynomial completion of  $X$  is

$$C_P(X) = \{y \in C^n \mid |f(y)| \leq \sup_{x \in X} |f(x)| \text{ for all polynomials } f\}.$$

Define  $C_M(X)$  to be the intersection of  $V$  and all sets of the form  $\{z \in C^n \mid |z_1^{m_1} \dots z_n^{m_n}| \leq r\}$  which contain  $X$ , where  $m_1, \dots, m_n \geq 0$ . The assertion is that  $C_P(X) = C_M(X)$ . An application to convergence sets of certain formal power series is also included.

R. C. Gunning.

**Ozaki, Shigeo; Kashiwagi, Sadao; and Tsuboi, Teruo.** On the Bloch's constant in several complex variables. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 5 (1956), 115-121.

Let  $z$  denote a  $k$ -dimensional column vector and  $e$  the  $k$ -dimensional identity matrix. The first main result is the following. If  $w = f(z) = a_0 + ez + a_2 z^2 + \dots$  is regular for  $|z| \leq R$  and satisfies the condition  $\|df(z)/dz\| \leq M$ , then  $w$  is univalent for  $|z| < R/M$ , and the image domain of  $|z| < R$  mapped by  $w = f(z)$  contains the univalent hypersphere with center  $w_0 = f(0) = a_0$  and radius

$$MR \left\{ 1 + (M^2 - 1) \log \frac{M^2 - 1}{M} \right\}.$$

(The norm  $|z|$  denotes the square root of  $z^* z$  where  $z^*$  is the conjugate row vector of  $z$ . The norm  $\|a\|$  of an  $n$ -dimensional square matrix means  $\sup \|ax\|/|x|$ , where  $x$  runs over all  $k$ -dimensional column vectors). The authors use this result to obtain a lower bound on the radius of a univalent hypersphere contained in the image of  $|z| < 1$  by suitable mappings of the form  $w = f(z) = z + a_2 z^2 + \dots$ .

W. T. Martin (Cambridge, Mass.).

**Ozaki, Shigeo; and Matsuno, Takeshi.** Note on bounded functions of several complex variables. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 5 (1956), 130-136.

The authors obtain the following theorem. Let  $W(Z)$  and  $Z$  be matrices of type  $(m, n)$ . If  $W(Z)$  is regular and if  $\|W\| < 1$  for  $\|Z\| < 1$ , then

$$\left\| \frac{dW}{dZ} \right\| \leq \frac{\|E_m - W^* W\| \|E_m - W W^*\|^{1/2}}{1 - \|Z\|^2} \leq \frac{1}{1 - \|Z\|^2},$$

where  $E_m$  and  $E_n$  are unit square matrices of order  $m$  and  $n$  respectively. They also obtain two sided inequalities on  $\sigma(W(Z))$  for the case of square matrices,  $m=n$ . Here

$\sigma(A)$  denotes the minimum positive square root of the characteristic values of  $A^*A$ . In the final section the authors obtain bounds on  $\|dW/dZ\|$ , where  $W(Z)$  is an  $m$ -column vector function of an  $m$ -column vector variable  $Z$ , and where it is assumed that  $|W| < 1$  for  $|Z| < 1$ .

W. T. Martin (Cambridge, Mass.).

**Sugawara, Masao. On the theory of kernel functions.**

J. Fac. Sci. Univ. Tokyo. Sect. I. 7 (1956), 265-303.

In the paper the author deals with matrices of type  $(m, n)$ . For example  $Z$  denotes the matrix with elements  $z_{jk}$  ( $j=1, \dots, m; k=1, \dots, n$ ). In the first part the author defines a kernel function in this matrix space and studies its properties. In the second part he discusses circular domains and movements of the unit-sphere (that is, one-one pseudoconformal mappings of the unit sphere onto itself). He then obtains the kernel function for the unit-sphere and uses it to define an invariant metric for the unit-sphere. He also discusses geodesic lines and curvature. In the paper the author makes use of earlier work of S. Bergman [Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques, Interscience, New York, 1941; The kernel function and conformal mapping, Math. Surveys, no 5, Amer. Math. Soc., New York, 1950; MR 2, 359; 12, 402] and C. L. Siegel [Amer. J. Math. 65 (1943), 1-86; MR 4, 242].

W. T. Martin.

See also: Eichler, p. 299; Tanaka, p. 301; Akutowicz, p. 304; Mandelbrojt, p. 305; Danilyuk, p. 311; Bellman, p. 330; Salzer, p. 339; Radok, p. 349.

**Geometrical Analysis**

**★Barrett, Leonard L. An introduction to tensor analysis.**

The National Press, Palo Alto, California, 1956. ii+33 pp. \$2.00.

This book is a very brief exposition of the elements of tensor analysis. Because of its brevity and lack of mathematical precision it will probably not appeal to the mathematical public.

C. B. Allendoerfer.

**Goido, Š. M. A double integral.** Grodnenskiĭ Gos. Ped. Inst. Uč. Zap. 1 (1955), 25-29. (Russian)

**Nevanlinna, Rolf. Über den Satz von Stokes.** Ann. Acad. Sci. Fenn. Ser. A. I. no. 219 (1956), 24 pp.

This is a continuation of work by the author [see, e.g., same Ann. nos. 169 (1954); 185 (1955); MR 15, 801; 16, 806] on the theory and application of vector calculus. This paper is concerned with the classical integral formula of Stokes. After some preliminaries concerning simplices in a (finite or infinite dimensional) real vector space  $L$ , the author introduces the integral of a continuous alternating form over a simplex  $s_n$ . If  $A(x)h_1 \cdots h_m$  is a real  $m$ -linear form which is differentiable for  $x$  in  $s_n$ , then the rotor of  $A$ ,  $\text{rot } Ah_1 \cdots h_m$ , is first defined as  $(-1)^m$  times the alternating part of the  $(m+1)$ -linear form  $A'(x)h_1 \cdots h_{m+1}$ . Two proofs are then offered for Stokes' formula

$$\int_{s_{m+1}} \text{rot } A = \int_{\partial s_{m+1}} A.$$

A second definition of  $\text{rot } A$  in terms of circulation density is then introduced and discussed briefly. Next the problem of solving the equation  $\text{rot } X = A$  in a convex region  $G$  of  $L$

is treated. Here the rotor is taken in the generalized sense and it is assumed that the alternating form  $A$  satisfies the conditions that  $A(x)$  is continuous in  $G$  and that for each  $x$  in  $G$ ,  $\text{rot } A$  exists. (The existence of the derivative  $A'(x)$  or the continuity of  $\text{rot } A$  is not assumed). It is proved that the stated equation has a solution if and only if  $\text{rot } A(x) = 0$  in  $G$ .

R. G. Bartle (Urbana, Ill.).

See also: Kneser, p. 192; Reade, p. 199; Nowacki, p. 247; Brinkman, p. 259; Eichler, p. 297; Guderley, p. 314; Toupin, p. 349; Durand, p. 362; Ouchi, Senba, and Yonezawa, p. 361; Iso and Kawaguchi, p. 361; Shibata, p. 362; Bergmann, p. 363.

**Functions with Particular Properties**

**Choquet, Gustave. Les noyaux réguliers en théorie du potentiel.** C. R. Acad. Sci. Paris 243 (1956), 635-638.

L'A. étudie en espace localement compact  $E$  les potentiels  $U^\mu(x) = \int G(x, y) d\mu(y)$  de mesures  $\mu > 0$  relatifs à des noyaux quelconques  $G \geq 0$  et énonce une série de définitions, résultats et exemples.  $G$  est dit régulier si pour toute  $\mu$  à support compact  $K$  la propriété que la restriction de  $U^\mu$  à  $K$  soit finie continue entraîne la même propriété de  $U^\mu$  dans  $E$ . On compare cette définition à quelques autres, en particulier à la "condition du maximum  $\lambda$ -dilaté pour tout compact". Cela signifie que pour tout compact  $X$ , il existe une constante  $\lambda(X) \geq 1$  telle que si  $\mu$  est de support  $K \subset X$ ,  $\sup_X U^\mu \leq \lambda \sup_K U^\mu$ . Cette condition entraîne la régularité de  $G$ . La réciproque n'a lieu que pour les compacts  $X$  ne rencontrant pas un certain ensemble discret, d'ailleurs vide dans des cas généraux importants.

M. Brelot (Paris).

**Green, John W. Mean values of harmonic functions on homothetic curves.** Pacific J. Math. 6 (1956), 279-282.

If  $U(P)$  is harmonic in the plane, the mean value of  $U$  over the circumference or the area of a circle with center  $P_0$  equals  $U(P_0)$ . These well-known facts may be expressed by saying that families of concentric circles have the following property: the perimeter or area average of  $U$  is the same for each member of the family. The author shows that, under suitable regularity assumptions, there are no other families of homothetic curves which have this property.

A. Edrei (Syracuse, N.Y.).

**Choquet, Gustave. Theory of capacities.** Ann. Inst. Fourier, Grenoble 5 (1953-1954), 131-295 (1955).

This paper forms the foundation of a new and abstract approach to potential theory in the spirit of Bourbaki, making extensive use of set theoretic, measure theoretic and topological methods. As such, it parallels, and forms a worthy companion to, the fundamental work on modern potential theory in the framework of Lebesgue measure and integration, the thesis of Frostman [Medd. Lunds Univ. Mat. Sem. 3 (1935)]. As the title implies, the principal results concern certain questions relative to the notion of capacity which the older theory was unable to answer. These center about the notion of "capacitability" of sets, i.e., the equivalence of inner and outer capacities, analogous to the concept of measurability. Certain preliminary results in this direction were given previously by the author [Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R.I., 1952, p. 376; C. R. Acad. Sci. Paris 232 (1951), 2174-

2176; 233 (1951), 904-906; 234 (1952), 35-37, 383-385, 498-500, 784-786; MR 13, 19, 633, 555, 829].

The problem originates with the consideration of the classical case of Borel sets in 3-dimensional Euclidean space relative to the usual Newtonian potential, which had previously been proposed by H. Cartan [Bull. Soc. Math. France 73 (1945), 74-106; MR 7, 447]. The author generalizes this immediately by the introduction of Borel and analytic sets in arbitrary topological spaces, defined, however, by countable union and intersection of compact rather than closed or open sets, and omitting complementation entirely. Thus the fundamental role is played by  $K_{\infty}$  sets rather than by  $G_{\delta}$  sets as in the classical case. It follows that if every open set is a  $K_{\sigma}$ , which is true for a separable and locally compact metric space, then these definitions are equivalent to the classical ones. Newtonian and "Greenian" capacities (i.e., relative to a domain possessing a Green's function) are then defined for arbitrary compact sets in Euclidean spaces in the usual manner in terms of the potential function  $U^{\mu}(Q) = \int G(P, Q) d\mu(P)$ , where  $G(P, Q)$  is the Green's function (if it exists) of the given domain  $D$ , and  $\mu$  is a Radon measure of a compact subset  $KCD$ . Certain well-known properties of these capacities are given, and inner and outer capacities of arbitrary sets are then defined from them in the usual manner. These capacities are considered from a set-theoretic point of view, and many interesting inequalities and extensions are developed, such as the introduction and consideration of alternating and monotone set functions and capacities, eventually culminating in a fundamental sequence of inequalities which imply that  $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$  for arbitrary compact sets  $A$  and  $B$  and for many capacities  $f$  including the classical Newtonian capacity in 3-dimensional Euclidean space. [Reviewer's Note: This inequality, even in a somewhat restricted form, does not hold for the logarithmic capacity in the plane [Lepson, Bull. Amer. Math. Soc. 61 (1955), 148-149]. The important question is still open, therefore, as to whether the basic results on capacitability hold for that case.] This inequality is then used to establish the capacitability of Borel and analytic sets for large classes of alternating capacities and spaces, including the above. By the use of some recent results of Gödel and Novikov [Novikov, Trudy Mat. Inst. Steklov. 38 (1951), 279-316; MR 14, 234] on the non-measurability in the sense of Lebesgue of certain projective sets on the real line, it is shown that these results cannot be generally extended to the complements of analytic sets. Analogous results are then obtained for monotone capacities.

The paper closes with a chapter in which these methods, combined with the well-known theorem of Kreĭn and Mil'man concerning convex and compact subsets of locally convex spaces, are applied to the study of the extremal elements of convex cones and integral representations of functions. This leads to a probabilistic interpretation of certain capacities, and is motivated by the analogy between set functions which are alternating of order infinity and functions of a real variable whose successive derivatives are alternately positive and negative. These latter "completely monotone" functions have been shown by S. Bernstein to have an integral representation in terms of the functions  $e^{-tx}$ .

B. Lepson (Washington, D.C.).

**Brelot, M.** Existence theorem for  $n$ -capacities. Ann. Inst. Fourier, Grenoble 5 (1953-1954), 297-304 (1955).

This paper establishes a result needed by Choquet in

the paper reviewed above. The author establishes the existence, in an arbitrary Green space, of  $n$  disjoint compact sets each of unit capacity such that their union has capacity arbitrarily close to  $n$ . This includes, in particular, Euclidean spaces and arbitrary hyperbolic Riemann surfaces [same Ann. 3 (1951), 199-263; MR 16, 34]. The result may also be extended to a finite number of compact sets of different capacities.

B. Lepson.

**Huber, Alfred.** On an inequality of Fejér and Riesz. Ann. of Math. (2) 63 (1956), 572-587.

L. Fejér and F. Riesz have shown [Math. Z. 11 (1921), 305-314] that for any conformal map of the unit circle onto a plane domain, the length of the map of the circumference is at least twice that of the map of any diameter. The reviewer [J. London Math. Soc. 13 (1938), 82-86] extended this complex-variable inequality to the case of an arbitrary function of class  $PL$  — that is, to any nonnegative function having a subharmonic logarithm — and applied it to the theory of surfaces of non-positive Gaussian curvature; regularity assumptions were later weakened by S. Lozinsky [Izv. Akad. Nauk SSSR. Ser. Mat. 8 (1944), 175-194; MR 6, 155].

The author now extends the inequality further, showing that if  $u(z)$  is the difference of two subharmonic functions,  $u(z) = u_1(z) - u_2(z)$  on the closed unit circle  $|z| \leq 1$ , and the measure  $\mu_2(e)$  associated with  $u_2(z)$  satisfies  $\mu_2(|q| < 1) = \alpha < 1$ , then the inequality

$$\int_{-\pi}^{\pi} e^{u(e^{i\theta})} d\theta \geq 2 \cos(\pi\alpha/2) \int_{-1}^1 e^{u(\rho e^{i\theta})} d\rho$$

holds for arbitrary  $\theta$ ,  $0 \leq \theta < 2\pi$ . The conditions under which the sign of equality holds are determined, and applications are made to meromorphic functions, Hilbert's inequality, and the theory of surfaces.

E. F. Beckenbach (Los Angeles, Calif.).

**Hong, Imsik.** On positively infinite singularities of a solution of the equation  $\Delta u + k^2 u = 0$ . Kodai Math. Sem. Rep. 8 (1956), 9-12.

L'auteur étend aux intégrales de l'équation (1)  $\Delta u + k^2 u = 0$  dans le plan ou dans l'espace l'existence, établie par G. C. Evans [Monatsh. Math. Phys. 43 (1936), 419-424] pour  $k=0$ , d'une solution hors d'un compact  $E$  de capacité nulle qui tendrait vers  $+\infty$  aux points de  $E$ . On reprend le raisonnement de Evans basé sur un potentiel de masses ponctuelles sur  $E$  et la propriété de la capacité comme diamètre transfini; mais on introduit à côté du noyau  $\log r$  ou  $1/r$  celui qui donne la solution fondamentale générale de (1). La réciproque est facile puisqu'une solution  $>0$  de (1) est surharmonique.

M. Brelot (Paris).

**Lisevič, L. N.** Almost periodic solutions of a hyperbolic system of linear differential equations with almost periodic coefficients. Dopovidi Akad. Nauk Ukrain. RSR 1956, 220-222. (Ukrainian. Russian summary)

In this paper the author gives the definition of an almost periodic Levitan function ( $N_1$ ,  $N_2$ -almost periodic functions) [Almost periodic functions, Gostehizdat, Moscow, 1953; MR 15, 700], in the case of two arguments and generalizes results due to Favard [Acta Math. 51 (1927), 31-81] and Levitan [Uspehi Mat. Nauk (N.S.) 2 (1947), no. 6(22), 174-214; MR 10, 293] for ordinary differential equations with almost periodic coefficients to linear hyperbolic systems of equations with almost periodic

coefficients in both variables and almost periodic initial conditions. No proofs. *František Wolf.*

**Džvarševili, A. G.** On a sequence of integrals. Soobšč. Akad. Nauk Gruzin. SSR 17 (1956), 297-302. (Russian)

Let  $\{g_n(x)\}$  be a sequence of functions of bounded variation  $V_{ab}[g_n(x)]$  on the interval  $[a, b]$ . The author proves that the following two conditions are equivalent: (1) There exists for every function  $f(x)$  integrable on  $[a, b]$  in the sense of Denjoy-Perron a constant  $M(f)$  such that  $\int_a^b f(x)g_n(x)dx < M(f)$  for all  $n$  and all subintervals  $[a, \beta]$  of  $[a, b]$ ; (2)  $V_{ab}[g_n(x)] < C$  for all  $n$ .

*F. A. Behrend (Melbourne).*

See also: Livingston and Lorch, p. 288; Alaci, p. 288; Matschinski, p. 288; Ohtsuka, p. 292; Nehari, p. 293; Chen, p. 303; Markus and Moore, p. 306; Pini, p. 312; Aronszajn and Smith, p. 319; Hofmann, p. 341; Radok, p. 349; Čankvetadze, p. 351; Power and Scott-Hutton, p. 353; Konorski, p. 357.

### Special Functions

**Turri, Tullio.** Sulle tabelle dei periodi degli integrali abeliani. Rend. Sem. Fac. Sci. Univ. Cagliari 26 (1956), 68-78.

**Eichler, Martin.** Zur Zahlentheorie der Quaternionen-Algebren. J. Reine Angew. Math. 195 (1955), 127-151 (1956).

This paper contains the arithmetic background for the important results of the author on the representation of modular forms in terms of  $\theta$ -series, which is reviewed below. Suppose that  $k$  is a totally real finite algebraic number field in which a finite set of valuations  $\{o_i\}$ , containing the Archimedean valuations of  $k$ , is singled out. Let  $i$  denote the elements of  $k$  which are integral for all valuations  $p$  not in  $\{o_i\}$ ; this ring  $i$  shall be termed the principal order of  $k$ . The author considers (unless otherwise mentioned) orders  $\mathfrak{J}$  (finite  $i$ -modules with 2, respectively, 4 linearly independent elements) in extensions of  $k$  which are either quadratic fields or quaternionic algebras  $Q$  in which the valuations  $o_i$  are ramified (as it is always the case for the Archimedean valuations) or inert. Special emphasis is placed on quaternionic orders. The level  $q$  of  $\mathfrak{J}$  is defined as the inverse  $i$ -ideal of the greatest common divisor of the  $k$ -norms of the elements in the complement of  $\mathfrak{J}$ . It is then shown by local considerations, which incidentally predominate in this paper, that for primes  $p$  dividing a squarefree  $q$  the  $p$ -component of  $\mathfrak{J}$  is either the unique maximal order of the  $p$ -adic limit  $Q_p$  (division algebra) or, to within isomorphisms, the ring of all  $2 \times 2$  matrices from  $i_p$  whose elements in the lower left corner are divisible by the prime  $p$  of  $i_p$ . Next ideals of  $Q$  with respect to  $\mathfrak{J}$  are defined as intersections of principal ideals relative to the completions  $\mathfrak{J}_p$  (a fact which is always satisfied for maximal orders); the ambiguous ideals and numbers of integral ideals with the same norm are determined as a preparation for the theory of the  $\zeta$ -function. Then all ideals with left and right orders of fixed squarefree level  $q$  form a groupoid. Let  $H$  be the number of left ideal classes (equivalence with respect to the non-zero elements of  $Q$ ), with the representatives  $\mathfrak{J} = \mathfrak{M}_1, \dots, \mathfrak{M}_H$  with respect to a fixed order  $\mathfrak{J}$  of level  $q$ . As in the case of maximal orders the

right orders  $\mathfrak{J}_r$  of the  $\mathfrak{M}_r$  are distributed over  $T$  types (sets of non-isomorphic orders) for whose determination the local study of ambiguous ideals is needed. Suppose now that the ideals  $\mathfrak{M}_r$  are labeled so that  $\mathfrak{J}_1, \dots, \mathfrak{J}_T$  belong to the distinct types and such that  $T_{\mu\nu}$  ( $\mu=1, \dots, H_r$ ) is a system of representatives of all classes of ambiguous  $\mathfrak{J}_r$ -ideals. Next denote by  $e_r$  the finite (because of the conditions on  $\{o_i\}$  and  $k$ ) index of the unit group of  $i$  in that of  $\mathfrak{J}$ . Then the measure  $M = \sum_{r=1}^T H_r/e_r$  for which an evaluation by means of the  $\zeta$ -function is given is introduced as a tool for the computation of  $H$  and  $T$  by means of traces of certain matrices. Now let  $\pi_{\mu\nu}(n)$  be the number of integral  $\mathfrak{J}_\mu$ -ideals whose norm in  $k$  equals  $n$  and which are left equivalent to  $\mathfrak{M}_\mu^{-1}\mathfrak{M}_\nu$ . Furthermore denote by  $L(n)$  the matrix with 1 at the position  $(\mu, \nu = \lambda(\mu))$  and 0 elsewhere, where  $\lambda(\sigma)$  is the permutation of  $1, \dots, H$  so that  $\mathfrak{M}_\sigma n$  is left equivalent to  $\mathfrak{M}_\tau$ ,  $\tau = \lambda(\sigma)$ . Then  $L(mn) = L(m)L(n)$ . Finally let  $E$  be the diagonal matrix of the indices  $e_r$ . Then the matrices  $P(n) = (\pi_{\mu\nu}(n))$  which were introduced by H. Brandt are shown to satisfy  $P(n)E = P(n)L(n)E' = EL(n)^{-1}P(n)'$ . Furthermore in analogy to the matrices of E. Hecke the following formulas are proved by ideal-theoretic methods:

$$P(m)P(n) = P(mn) \text{ if } (m, n) = 1,$$

$$P(p^a)P(p^b) = \sum_{r=0}^b N(p)P(p^{a+b-2r})L(p^{-1})^r \quad (a \geq b, p \nmid q)$$

(a discussion for certain  $p|q$  is also given). As a consequence of these facts the Dirichlet series  $\sum_n P(n)/N(n)^s$  has an Euler product for which almost all  $p$ -factors are canonical according to Hecke's definition. After these important preliminary results the author turns to the proof of the important theorem which states that each (commutative) ring which is generated over the field of real numbers by matrices  $P(n)$  and  $L(n)$  must be semisimple. This result is of special significance for the structure of the multiplication rings of certain fields of modular functions (see the review below). If  $Sp$  denotes the trace, the  $H = Sp(P(i))$  and  $T = 2^{-\kappa} \sum_t Sp(P(t))$ , where  $\kappa$  is the number of prime divisors  $t$  of  $q$ . In turns the traces  $Sp(P(n))$  are evaluated in terms of the measure  $M$  and a rather complicated correction term whose nature depends on certain quadratic subfields of  $Q/k$  (see Theorem 10, p. 143). The author terminates the paper with some special cases which yield among other noteworthy relations between the class numbers of different imaginary quadratic number fields. Furthermore the generalization of the theory to function fields of Kroneckerian dimension 1 is expounded. *O. F. G. Schilling (Chicago, Ill.).*

**Eichler, Martin.** Über die Darstellbarkeit von Modulformen durch Thetareihen. J. Reine Angew. Math. 195 (1955), 156-171 (1956).

Suppose that  $K$  is the field of all modular functions  $\varphi(\tau)$  over the field of complex numbers  $k$  which are invariant with respect to the congruence group of unimodular transformations  $\Gamma_0(q) = \{\tau \rightarrow (a\tau + b)/(qc\tau + d); a, b, c, d \text{ being integers, } q \text{ a squarefree integer}\}$ . Denote by  $g$  the genus of  $K$ . Furthermore let  $\begin{pmatrix} s & r \\ 0 & t \end{pmatrix}$  with  $st = n$ ,  $(n, q) = 1$ ,  $s > 0$ ,  $(r, s, t) = 1$ ,  $1 \leq r \leq t$  denote a complete set of representatives for the left cosets of primitive matrices  $\begin{pmatrix} a & b \\ qc & d \end{pmatrix}$ ,  $a, \dots, d$  again being integers, with determinant  $n$ . Next suppose that  $\sigma_i(\tau)$ ,  $1 \leq i \leq g$ , is a basis for the vector space of cusp forms of weight  $-2$  which belong to the

group  $\Gamma_0(q)$ . Then each sum

$$\sum_{r,s,t} (s/t) \sigma_t \left( \frac{sr+t}{t} \right) d\tau = (\sigma_t(\tau) | \bar{\tau}_n) d\tau = \sum_{k=1}^q \bar{t}_{tk}(n) \sigma_k(\tau) d\tau$$

is with  $\sigma_t(\tau) d\tau$  a differential of the first kind of  $K$ , and thus  $\tau_n = \sum_{r \in \mathbb{N}} \bar{\tau}_{nr} \tau_r$  defines the operator of Hecke and a modified operator  $\tau_n^0 = \sum_{r \in \mathbb{N}} \bar{\tau}_{nr} \tau_r$ . Using the matrices  $T_n = (t_{tk}(n))$  representations  $T_n$  and  $T_n^0$  of  $\tau_n$  and  $\tau_n^0$ , respectively, are obtained. The author proves in this paper the validity of Hecke's conjecture that the entire modular forms of weight  $-2$  belonging  $\Gamma_0(q)$ ,  $q$  a prime, are linear combination of  $\theta$ -series defined by quaternary quadratic forms of level  $q$  and discriminant  $q^2$ . For the proof the author considers for arbitrary squarefree  $q$  the functions

$$\theta_{\mu\nu} = \sum_{M \in \mathbb{N}} \exp \left( \frac{n(M)\tau}{n(\mathbb{N}) - i\mathbb{N}\mu} \right) = 1 + \sum_{n=1}^{\infty} \pi_{\mu\nu}(n) \omega_{\nu} \exp(n\tau)$$

which are integral modular forms of weight  $-2$  belonging to  $\Gamma_0(q)$ . The author then forms the matrix

$$\theta(\tau) = \sum_{n=0}^{\infty} P(n) \exp(n\tau),$$

where  $P(0)$  is the matrix with the rows  $(1/w_1, \dots, 1/w_H)$  with  $w_r$  the number of units in the order  $\mathfrak{F}_r$  according to the definitions of the preceding review. Application of Hecke's operator  $\tau_p$ ,  $p$  a prime not dividing  $q$ , yields

$$\theta(\tau) | \tau_p = \sum_{n=0}^{\infty} \sum_{r,s,t} P(n) \exp \left( n \left( \frac{sr+t}{t} \right) \right) = \sum_{n=0}^{\infty} \left( P(n\phi) + pP \left( \frac{n}{p} \right) \right) \exp(n\tau),$$

where  $st = \phi$ ,  $s > 0$ ,  $0 \leq r < p$ . Since the matrices  $P(n)$  (see the above review) satisfy the same relations as Hecke's  $\tau_n$ ,  $(n, q) = 1$ , one obtains in general  $\theta(\tau) | \tau_n = P(n) \theta(\tau)$ . Using the fact that the differences of such quaternary  $\theta$ -series are cusp forms as follows from the proof of their functional equations it is shown that the representation  $P(n)$  splits into one belonging to the Eisenstein series

$$c_0 + \sum_{n=1}^{\infty} \sum_{t|n, (t,q)=1} t \exp(n\tau)$$

and into another one of dimension  $H-1$ ,  $P_{q_1, q_2}(n)$  which is obtained in the subspace of cusp forms generated by  $\theta$ -series. The integers  $q_1, q_2$  are defined by  $q = q_1 q_2$ , where  $q_2$  is the product of the odd number of finite ramifications of a positive definite quaternionic algebra. In the above mentioned paper the author evaluated the trace  $\text{Sp}(P_{q_1, q_2}(n))$  in terms of Legendre symbols, class and unit numbers of certain imaginary quadratic orders which depend on  $n$  and on the factorization of  $q$ . Furthermore the author uses the fact that the matrix  $T_n^0$  is a representation of  $\tau_n^0$  considered as a correspondence of  $K$ . The latter is defined in the composite of  $K$  with an isomorphic field  $K'$  relative to a  $\tau'$ -plane by the ideal  $\mathfrak{I}_n^0$  which is generated in the subring of all finite sums  $\sum \varphi_r \varphi_r', \varphi_r \in K, \varphi_r' \in K'$ , by all products

$$\prod_{r,s,t} \left( \varphi(\tau) - \varphi \left( \frac{sr+t}{t} \right) \right),$$

where  $\varphi$  varies over all functions of  $K$ . A result of Hurwitz then states that  $\text{Sp}(T_n^0) = (d_n^0 + d_n^{0'} - c_n^0)$ , where  $d_n^0$  and  $d_n^{0'}$  denote the degree of the divisors belonging to  $\mathfrak{I}_n^0$  in the field composite relative to  $K'$  and  $K$ , respectively. One finds as a simple consequence of the definition that  $d_n^0 = d_n^{0'} = \sum_{t|n} t - x =$  the number of prime factors

of  $\mathfrak{I}_n^0$ , where  $x=1$  if  $n$  is a square and  $x=0$  otherwise. Finally  $c_n^0$  is the number of coincidence of  $\mathfrak{I}_n^0$ . This number being the sum of local coincidence numbers  $h_p$  for all prime divisors  $p$  of  $K$  is then split into a term involving only the rational cusps of a canonical fundamental domain of  $\Gamma_0(q)$  which give the total contribution  $c_{n, \infty}^0$  and a remainder term  $c_{n, 0}^0$ . These cusps are represented by the  $2x$  points  $\tau = \infty, 0, 1/q_1$ , where  $q_1$  denotes a typical non-trivial divisor of  $q$ . Localization of  $\mathfrak{I}_n^0$  to the corresponding prime divisor of  $K$  then yields by means of a matrix theoretic reduction to the case  $\tau = \infty$  that these local multiplicities coincide and equal  $2 \sum_{t|n, t < \sqrt{n}} t + y$ , where  $y = \sqrt{n} - 1$  if  $n$  is a square and  $y = 0$  otherwise. The evaluation of  $c_{n, 0}^0$  is quite complicated, however not for the contributions at the elliptic boundary points of the fundamental domain where

$$\frac{1}{2} \prod_{p|q} \left( 1 + \left( \frac{-4}{p} \right) \right) + \frac{2}{3} \prod_{p|q} \left( 1 + \left( \frac{-3}{p} \right) \right) = \alpha$$

is obtained. The requirement that  $\tau_0$  in the upper half plane be an inner point of the fundamental domain of  $\Gamma_0(q)$  and have positive coincidence number can be expressed in the form

$$\frac{A\tau_0 + B}{qC\tau_0 + D} = \tau_0, \det \begin{pmatrix} A & B \\ qC & D \end{pmatrix} = n, (A, B, C, D) \neq \sqrt{n},$$

$$(*) \quad N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = u + vU, U \in \Gamma_0(q),$$

$u, v$  rational (the latter inequality expresses that  $\tau_0$  is not elliptic). The author then interprets  $N$  as the generator of an imaginary quadratic order  $\mathfrak{D}$  with the discriminant  $\Delta f^2$ , where  $\Delta = \Delta(N) = \sigma^2 - 4n$ ,  $A + D = \sigma$ . Next let  $c(\Delta)$  denote the number of classes (with respect to  $\{U\} = \Gamma_0(q)$ ) of orders  $U^{-1}\mathfrak{D}U$  which can be optimally imbedded in  $\mathfrak{D} = \left\{ \begin{pmatrix} a & b \\ qc & d \end{pmatrix}, a, \dots, d \text{ integers} \right\}$  (optimal imbedding means that the intersection of  $\mathfrak{D}$  with the quotient field of the quadratic order equals the given order). Since the above  $\tau_0$  have the coincidence number 1 the desired contribution  $\beta$  equals the number of classes of substitutions  $U^{-1}NU$ ,  $U \in \Gamma_0(q)$ , which satisfy the condition (\*). It follows that  $\beta = \sum_{\sigma, f} c(\Delta f^2)$  for all  $\sigma, f$  which obey (†)  $-2\sqrt{n} < \sigma < 2\sqrt{n}$ ,  $0 < f$ ,  $\Delta f^2 \equiv 0$  or  $1 \pmod{4}$ ,  $\Delta f^2 < -4$ . Finally the arithmetic of the order  $\mathfrak{D}$  (for the methods used see also the paper reviewed above) implies that

$$c(\Delta) = \prod_{p|q} \left( 1 + \left\{ \frac{\Delta}{p} \right\} \right) h(\Delta),$$

where  $h(\Delta)$  denotes the class number of the order  $\mathfrak{D}$  and  $\{ \cdot \}$  stands for the Legendre symbol  $\left( \frac{\Delta}{p} \right)$  if  $\Delta p^{-2} \not\equiv 0$  or  $1 \pmod{4}$  and for 1 otherwise. Combining the above contributions the author obtains

$$c_{n, 0}^0 = \sum_{\sigma, f} \prod_{p|q} \left( 1 + \left\{ \frac{\Delta f^2}{p} \right\} \right) h(\Delta f^2) / w(\Delta f^2) - \gamma_{n, 0},$$

where (i) the summation is taken over all  $\sigma, f$  obeying (†) except the last inequality, and (ii)  $\gamma_{n, 0}$  equals 0 if  $n$  is not a square and  $\alpha$  if  $n$  is a square, (iii)  $w(\Delta)$  equals half the number of units in the corresponding order  $\mathfrak{D}$ , this latter correction factor is needed since the condition  $\Delta < -4$  is abandoned. Combination of these facts and the results of the author on quaternionic algebras yields ultimately, noting that  $\mathfrak{I}_n = \mathfrak{I}_n^0$  if  $n$  is not a square and  $\mathfrak{I}_n = \mathfrak{I}_n^0 +$  the identity if  $n$  is a square, that  $\text{Sp}(\mathfrak{I}_n) = \text{Sp}(P_{q, 1}(n))$  for  $(n, q) = 1$  if  $q$  is a prime, by virtue of a class number

relation. This equality of traces implies that the representation of the ring of multiplications on  $K$  which is generated by the  $\tau_n$  with  $(n, q) = 1$  on the space of cusp forms and the subspace generated by the  $\theta$ -series, respectively, are equivalent. This fact in turn implies that these spaces have the same dimension and that each cusp form equals a linear combination of  $\theta$ -series plus a form whose first  $q-1$  Fourier coefficients are 0, and hence by a theorem of Hecke the author's result is reached. Furthermore the author exhibits, using again his method of comparing traces, a whole class of cusp forms which cannot be obtained by means of  $\theta$ -series. Finally it is proved that the entire modular forms of weight  $-2k$ ,  $k=1, 2, \dots$ , belonging to  $J_0(q)$  are linear combinations of  $4k$ -fold  $\theta$ -series belonging to the levels 1 and  $q$  provided the prime  $q$  is sufficiently large. The proof is achieved by observing that the field  $K$  is not hyperelliptic for large  $q$  and that hence each integral differential  $\varphi(\tau)(d\tau)^k$  of degree  $k$  is a homogeneous polynomial of degree  $k$  of differentials of the first kind. *O. F. G. Schilling.*

**Eichler, Martin.** On the class number of imaginary quadratic fields and the sums of divisors of natural numbers. *J. Indian Math. Soc. (N.S.)* 19 (1955), 153-180 (1956).

In this paper the author applies the arithmetic results of the two papers reviewed above, in particular those concerning the number of fixed points of Hecke's correspondences  $T_n$  ( $n$  odd) on the Riemann surface  $R$  of genus 0 belonging to the modular group of level 2 (consisting of the matrices  $\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}$  of determinant 1,  $a, \dots, d$  being integers) to problems of class number relations for imaginary quadratic orders. For every negative  $D \equiv 0, 1 \pmod{4}$  there exists a quadratic ring  $\mathcal{O}(D)$  with  $D$  for its discriminant (generated over the integers by  $\sqrt{D}$  or  $\sqrt{D/4}$ , respectively). Then denote by  $h(n)$  the number of ideal classes in  $\mathcal{O}(-n)$  for  $n \equiv 0, 3 \pmod{4}$  (ideal classes defined as in the above papers),  $n > 4$ , and set  $h(0) = -1/12$ ,  $h(3) = 1/3$ ,  $h(4) = 1/2$ , and  $h(n) = 0$  otherwise. The author shows how the function  $H(\tau) = \sum_{n=0}^{\infty} h(n) \exp(n\tau)$  is related to  $\theta$ -functions of one variable, the discriminant function  $\Delta(\tau)$  of the theory of modular functions and other "elementary" functions. Thus let  $\varphi(\tau) = \sum_{n=1}^{\infty} \lambda(n) \exp(n\tau)$ , where  $\lambda(n) = \sum_{d|n, d < \sqrt{n}} d + \frac{1}{2} \sqrt{n}$  for  $n$  a square, and  $= \sum_{d|n, d < \sqrt{n}} d$  for all other  $n$ . Let

$$g(\tau) = -\frac{1}{48\pi i} \frac{d \log \Delta(\tau/2)}{d\tau/2}.$$

The first result of the author is the formula

$$H(\tau)\theta_{00}(\tau) = 2g(4\tau) - 2\varphi(4\tau) + \frac{1}{2}[g(\tau) - g(\tau+1)] - \frac{1}{2}[\varphi(\tau) - \varphi(\tau+1)]$$

which implies the known class number relation

$$\sum_{0 \leq s \leq 4m} h(4m-s^2) = 2\lambda(n)$$

for  $n=4m$ , and the new result

$$\sum_{0 \leq s \leq n} h(n-s^2) = \frac{1}{2} \sum_{d|n} d - \sum_{d|n, d < \sqrt{n}} d$$

for odd  $n$  where the sum on the left is further restricted by  $s \equiv \frac{1}{2}(n+1) \pmod{2}$ . These class number formulas are proved directly by interpreting them in terms of fixed point numbers (see the paper reviewed second above), using Lefschetz fixed point formula for  $R$  and  $T_n$ , and in terms of the number of integral ideals with given norm in a maximal order of the ordinary quaternions. The formula

for  $H(\tau)$  which is thus obtained by a combination of arithmetic and topological results is then used to establish the new class number relations

$$\sum_{0 \leq s \leq 4n} (s^2 - n)h(4n-s^2) = -2 \sum_{d|n, d < \sqrt{n}} d^3$$

which are equivalent to the formula

$$\begin{aligned} & \frac{1}{\pi i} [H(\tau)\theta_{00}'(\tau) - \frac{1}{2}(H(\tau)\theta_{00}(\tau))'] \\ & + \frac{1}{\pi i} [H(\tau+1)\theta_{00}'(\tau+1) - \frac{1}{2}(H(\tau+1)\theta_{00}(\tau+1))'] = \\ & -4 \left[ \chi(4\tau) - \frac{1}{\pi i} \varphi'(4\tau) \right], \end{aligned}$$

where

$$\chi(\tau) = \sum_{n=1}^{\infty} \left( \sum_{d|n, d < \sqrt{n}} d^3 + n\tau \right) \exp(n\tau),$$

the asterisk indicating that  $t = \sqrt{n}$  is counted with the multiplicity  $1/2$ . This formula is proved by ingenious analytic arguments using contour integrals for suitably defined auxiliary functions and their behavior at  $\tau = \infty, 0$ . A final inference of the above formula is the interesting functional equation

$$\begin{aligned} H(\tau) + (-i\tau)^{-3/2} H(-1/\tau) = \\ \frac{-1}{24} \theta_{00}(\tau)^3 - i^{1/2} \frac{\tau}{4} \int_{-\infty}^{\infty} \frac{1 + \exp(\tau\xi)}{1 - \exp(\tau\xi)} \exp(-\tau\xi^2) \xi d\xi. \end{aligned}$$

*O. F. G. Schilling* (Chicago, Ill.).

**Eichler, Martin.** Berichtigung zu der Arbeit "Über die Darstellbarkeit von Modulformen durch Thetareihen". *J. Reine Angew. Math.* 196 (1956), 155.

The author corrects a detail of the proof for the evaluation the multiplicity number  $c_{n,0}$  (see the second review above; the order which is determined by the substitution  $N$  does in turn determine is general a second substitution from which it may arise).

*O. F. G. Schilling* (Chicago, Ill.).

**Klingen, Helmut.** Diskontinuierliche Gruppen in symmetrischen Räumen. I, II. *Math. Ann.* 129 (1955), 345-369; 130 (1955), 137-146.

Continuing the work of Siegel [*Amer. J. Math.* 65 (1943), 1-86; MR 4, 242] and Braun [*Ann. of Math.* (2) 53 (1951), 143-160; MR 12, 482], the author studies in I the first three types of irreducible bounded symmetric domains:

$$H_1: \frac{1}{2i} (Z - \bar{Z}) > 0; H_2: \frac{1}{2i} (Z - \bar{Z}) > 0, Z'Z = -E;$$

$$H_3: \frac{1}{2i} (Z - \bar{Z}) > 0, Z' = Z;$$

where  $Z$  is an  $n \times n$  matrix and  $\bar{Z}$  is the conjugate complex transposed matrix. He characterizes the group of one-to-one analytic mappings of  $H_v$  on itself ( $v=1, 2, 3$ ), studies their geometric properties under a Riemannian metric, and treats discontinuous groups of mappings. He shows that the mappings of  $H_2(H_3)$  on itself can always be extended to  $H_1$  with one exception ( $n=4, H_2$ ). Every discontinuous group of mappings of  $H_v$  on itself possesses a fundamental region with the usual properties.

II is devoted to examples of discontinuous groups and their fundamental regions. The groups, which are of the first kind, are developed by number-theoretic means, involving the group of units of certain Hermitian forms. The fundamental regions are then constructed with the

help of the Humbert reduction theory of positive definite forms in an algebraic number field. Among the groups treated is the modular group.  
*J. Lehner.*

**Steel, W. H.; and Ward, Joan Y. Incomplete Bessel and Struve functions.** Proc. Cambridge Philos. Soc. 52 (1956), 431-441.

The functions discussed are defined in terms of the real and imaginary parts of  $P_\nu(x, \omega)$  multiplied by  $x^\nu/\Gamma(\nu+1)2^\nu$ , where

$$P_\nu(x, \omega) = \frac{2\Gamma(\nu+1)}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})} \int_0^1 e^{i\pi t} (1-t^2)^{\nu-1/2} dt, \quad \nu > -\frac{1}{2}.$$

They are important in the diffraction theory of optical instruments. Recursion formulas, differential equations, series representations, asymptotic expansions, and special values for these functions are given as well as tables for  $P_0(x, \omega)$  and  $P_1(x, \omega)$ . {For  $\nu!$  read  $\Gamma(\nu+1)$ .}

*N. D. Kazarinoff (Ann Arbor, Mich.).*

**Allen, E. E. Polynomial approximations to some modified Bessel functions.** Math. Tables Aids Comput. 10 (1956), 162-164.

The following Bessel functions are approximated by polynomials of from 7 to 9 terms having coefficients of about the same order of magnitude.

$$I_0(x), I_1(x)/x, |x| \leq 3.75,$$

$$I_0(x)x^{1/2}e^{-x}, I_1(x)x^{1/2}e^{-x}, |x| \geq 3.75,$$

$$K_0(x) + \log(\frac{1}{2}x)I_0(x), K_1(x) - \log(\frac{1}{2}x)I_1(x), 0 < x \leq 2,$$

$$K_0(x)x^{1/2}e^x, K_1(x)x^{1/2}e^x, |x| \geq 2.$$

The accuracy is to about 7 or 8 significant figures. The results were obtained by the straight forward fitting of polynomials to tabulated data.  
*D. H. Lehmer.*

**Mitrinovich, Dragoslav S. Nouvelles formules relatives aux polynomes de Legendre.** C. R. Acad. Sci. Paris 243 (1956), 1387-1389.

The inequality

$$|d^s P_n/dx^s| \leq s! 2^{-s} \binom{n}{s} \binom{n+s}{s} \quad (n=1, 2, \dots, s \leq n; -1 \leq x \leq 1),$$

is given along with two of its consequences. The integral  $\int_{-1}^{+1} d^r P_m/dx^r d^s P_n/dx^s dx$  is evaluated.

*N. D. Kazarinoff (Ann Arbor, Mich.).*

**Parodi, Maurice. Equations de Mathieu et équations intégrales de Volterra.** C. R. Acad. Sci. Paris 243 (1956), 1006-1007.

The note indicates that by using Laplace transforms it can be shown that the solution of the Mathieu equation

$$y'' + (\eta + \gamma \cos t)y = g(t)$$

is equivalent to that of the Volterra integral equation

$$y(t) + \frac{\gamma}{\eta^{1/2}} \int_0^t \sin \eta^{1/2}(t-\tau) \cos \tau y(\tau) d\tau = G(t),$$

where  $G(t)$  depends on  $y'(0)$ ,  $y(0)$  and  $g(t)$ . A similar procedure is obviously applicable to the equation of Hill:  $y'' + (\eta + F(t))y = 0$ ,  $F(t)$  periodic, period  $2\pi$ , and to the partial differential equation

$$\partial^2 y / \partial t^2 + (\eta + \gamma \cos x) \partial^2 y / \partial x^2 = 0.$$

*T. H. Hildebrandt (Ann Arbor, Mich.).*

**Boas, R. P., Jr.; and Buck, R. C. Polynomials defined by generating relations.** Amer. Math. Monthly 63 (1956), 626-632.

The polynomials  $p_n(z)$  generated by  $A(w)\psi(zg(w))$ , where  $A(w)$ ,  $\psi(t)$ , and  $g(w)$  are convergent or formal power series, are described by means of an explicit formula and a recursion relation. It is proved that if  $\psi(t) = \sum_{n=0}^{\infty} \gamma_n t^n$ ,  $g(w) = \sum_{n=0}^{\infty} b_n w^n$ , and  $A(0) \neq 0$ , then the coefficients of  $w^n$  in the power series for  $A(w)\psi(zg(w))$  are polynomials if, and only if,  $b_0 = 0$  and of degree  $n$  if, and only if,  $b_1 \gamma_n \neq 0$ . The various classes of polynomials generated in this way include, among others, the Appell, Sheffer, and Brenke polynomials. A characterization by properties of the generating function leads, of course, to the recursion formula mentioned above. Given any sequence  $\{p_n(z)\}$ ,  $p_n$  of degree  $n$ , this characterization is: A necessary and sufficient condition that

$$K(z, w) = \sum_{n=0}^{\infty} w^n p_n(z) = A(w)\psi(zg(w))$$

with  $b_0 = 0$ ,  $b_1 = 1$ , and  $A(0) \neq 0$  is that there exist power series  $\alpha(w) = \sum_{n=0}^{\infty} \alpha_n w^n$  and  $\beta(w) = 1 + \sum_{n=1}^{\infty} \beta_n w^n$  such that  $K_w(z, w) = \alpha(w)K(z, w) + zw^{-1}\beta(w)K_z(z, w)$ . A set of  $p_n(z)$  closely related to the Appell polynomials is discussed.

*N. D. Kazarinoff (Ann Arbor, Mich.).*

See also: Bellman, p. 286; Schwartz, p. 288; Livingston and Lorch, p. 288; Komatu, p. 292; Hayman, p. 293; Eichler, p. 350; Erugin, p. 307; Kovalenko, p. 349; Kornecki, p. 350; Vekua, p. 350; Życzkowski, p. 352; Monin, p. 354; Steel, p. 354; Parkus, p. 359; Havlíček, p. 365.

### Sequences, Series, Summability

**Walsh, C. E. A note on convergence factors.** Proc. Edinburgh Math. Soc. (2) 9 (1956), 154-156.

Zu einer Zahlenfolge  $u_n \geq u_{n+1} > 0$  ( $n=1, 2, \dots$ ) werden Folgen  $e_1, e_2, \dots$  betrachtet, für die  $\sum_{n=1}^{\infty} u_n e_n$  ein gewisses asymptotisches Verhalten besitzt (etwa  $\sum_{n=1}^{\infty} u_n e_n = S + o(1)$ ). J. Karamata [J. London Math. Soc. 21 (1946), 162-166; MR 8, 456, 709] gab Bedingungen für die Folge  $\{u_n\}$  an, aus denen stets  $\sum_{n=1}^{\infty} e_n = o(n)$  folgt. Es handelt sich dabei um Verallgemeinerungen früherer Überlegungen von H. Rademacher [Math. Z. 11 (1921), 276-288] und W. H. J. Fuchs [Proc. Edinburgh Math. Soc. (2) 7 (1942), 27-30; MR 4, 79]. In der vorliegenden Arbeit werden die Überlegungen von Karamata so weitergeführt, dass auf  $\sum_{n=1}^{\infty} e_n = o(A_n)$  mit  $A_n = \sum_{r=1}^n a_r$ ,  $a_r \geq 0$ ,  $\sum a_r = \infty$ ,  $A_{n+1} = O(A_n)$  geschlossen werden kann.  
*A. Peyerimhoff.*

**Hatcher, J. R. A singular integral equation containing a parameter.** Amer. Math. Monthly 63 (1956), 651-652.

It is shown that if  $L$  (consisting of a finite number of smooth, non-intersecting contours of finite length) is the boundary of a plane connected region,  $f(t)$  and  $g(t)$  satisfy the Hölder condition on  $L$ , and  $\lambda$  is a parameter, then the integral transforms

$$(\pi i)^{-1} \int_L \frac{\cos \lambda(t-t_0)}{t-t_0} g(t) dt = f(t_0),$$

$$(\pi i)^{-1} \int_L \frac{\cos \lambda(t-t_0)}{t-t_0} f(t) dt = g(t_0),$$

where  $t_0 \in L$ , are reciprocal.

*L. B. Ball.*

Nilov, G. N. Expansion of a continuous function into a generalized power series. *Kabardinskii Gos. Ped. Inst. Uč. Zap.* 8 (1955), 25-27. (Russian)

Tanaka, Chuji. Note on Dirichlet series. XVI. On Borel-curves of the integral functions defined by Dirichlet series. *Yokohama Math. J.* 3 (1955), 127-140.

This note connects with Nos. XII-XIII of the same series [*Proc. Japan Acad.* 30 (1954), 157-159, 257-261; MR 16, 125] where the author considered order directions for an entire function  $F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s)$ , where the defining Dirichlet series is supposed to have its abscissa of uniform convergence equal to  $-\infty$ . The order of  $F(s)$  is defined by  $\limsup_{\sigma \rightarrow +\infty} (-\sigma)^{-1} \log^+ \log^+ M(\sigma) = \rho < \infty$  with  $M(\sigma) = \sup_t |F(\sigma + it)|$ . A Jordan curve  $C_B$  extending from  $+\infty$  to  $-\infty$  is a Borel curve for  $F(s)$  if for every  $\epsilon > 0$  and every  $a$ , with at most two exceptions, the zeros of  $F(s) - a$  in the strip  $D(\epsilon, C_B)$  swept out by the circular disc  $|s - s_0| \leq \epsilon$ ,  $s_0 \in C_B$ , satisfy the condition that  $\sum_{n=1}^{\infty} \exp[\alpha \Re\{s_n(a)\}]$  converges for  $\alpha > \rho$  and diverges for  $\alpha < \rho$ . Let  $L$  be the straight line  $\sigma \cos \theta + t \sin \theta = \rho$ ,  $|\theta| \leq \frac{1}{2}\pi$ ,  $\theta \neq 0$ . Let  $C = C(r, \theta, \rho)$  be a Jordan curve in the strip  $D(r, L)$ . Suppose there exists an  $R$  such that  $F(s)$  is of order  $\rho$  also in the strip  $D(R, C)$ . Then if  $R > \varphi(r, \theta, \rho)$  there exists at least one Borel curve in the strip  $D(R, C)$ . Using his previously established results on order directions, the author is able to prove the existence of Borel curves under suitable restrictions on  $\lambda_n$  and  $a_n$ .

E. Hille (New Haven, Conn.).

Wynn, P. Central difference and other forms of the Euler transformation. *Quart. J. Mech. Appl. Math.* 9 (1956), 249-256.

This paper develops modifications of Euler's transformation for the summation of series of form  $\sum (-1)^r a^r v_r$  in terms of central and mean central differences. An interesting feature is the construction of a lozenge diagram, similar to that devised by Duncan C. Fraser for interpolation formulae, which covers transformation formulae involving forward, central or backwards differences or a mixture of these. Numerical illustrations are given.

J. C. P. Miller (Cambridge, England).

Gagaev, B. M. Existence theorems for solutions of integrodifferential equations. *Uč. Zap. Kazan. Univ.* 115 (1955), no. 14, 21-28. (Russian)

The author considers the system of integrodifferential equations

$$\frac{d^{m_s} y_s(x)}{dx^{m_s}} = L_s(x, y_1(x), \dots, y_n(x)) + \lambda \int_a^b K_s(x, u) M_s(u, y_1(u), \dots, y_n(u)) du \quad (1 \leq s \leq n),$$

where  $L_s$  and  $M_s$  are differential operators, not necessarily linear, of order less than  $m_s$ , and seeks solutions  $y_1(x), \dots, y_n(x)$  satisfying the initial conditions

$$y_s(a) = \alpha_s, \quad y_s^{(r)}(a) = \beta_{sr} \quad (1 \leq r < m_s, 1 \leq s \leq n).$$

By a preliminary transformation he reduces the problem to that of finding solutions  $z_1(x), \dots, z_N(x)$  of a system of the form

$$\frac{dz_s}{dx} = P_s(x, z_1(x), \dots, z_N(x)) + \lambda \int_0^1 G_s(x, u) Q_s(u, z_1(u), \dots, z_N(u)) du \quad (1 \leq s \leq N),$$

where the expressions  $P_s$  and  $Q_s$  no longer contain derivatives; the initial conditions are now

$$z_1(0) = \dots = z_N(0) = 0.$$

Under the hypotheses that (i)  $P_s$  and  $Q_s$  are continuous for  $0 \leq x \leq 1$ ,  $|z_s(x)| \leq C$ ,  $|P_s| \leq B$ ,  $|Q_s| \leq B$ , (ii)  $G_s(x, u)$  is the kernel of an integral equation for which the Fredholm theorems hold and

$$\int_0^1 |G_s(x, u)| du \leq D,$$

and (iii)  $B(1 + |\lambda|D) \leq C$ , the author shows that the given system has at least one solution. This solution is unique if appropriate Lipschitz conditions are assumed to hold. The existence proof can be extended to certain cases in which  $P_s$  and  $Q_s$  are only measurable with respect to  $x$ .

The principal tool used in the proof of the main result is the Markov-Kakutani fixed-point theorem.

F. Smithies (Cambridge, England).

Evans, Arwel. A theorem on general regular transformations of series. *Proc. Edinburgh Math. Soc.* (2) 9 (1956), 105-108.

Es werden Transformationen  $t_m = f_m(s_1, \dots, s_m)$  von (reellen) Folgen in (reelle) Folgen betrachtet; dabei soll  $f_m$  erklärt sein für alle Folgen  $a < s_n < b$ . Hinreichend für die Regularität in  $(a, b)$  (d.h.  $s_n \rightarrow s$ ,  $a < s < b$ , impliziert  $t_m \rightarrow s$ ) sind die Bedingungen (i)  $f_m(s_1, \dots, s_k, x, \dots, x) \rightarrow x$  ( $m \rightarrow \infty$ ) für beliebige Zahlen  $a < x < b$ ,  $k$  und  $s_1, \dots, s_k$ ; (ii)  $f_m(s_1, \dots, s_m)$  ist für jedes  $m \geq N$  ( $N$  unabhängig von  $m$ ) eine monoton wachsende Funktion des Argumentes  $s_n$ ,  $n \geq N$ . Dieses Ergebnis stellt eine Verallgemeinerung des hinreichenden Teiles des Satzes von Toeplitz bei positiven Dreiecksmatrizen dar.

A. Peyerimhoff (Giessen).

Zeller, Karl. Vergleich des Abelverfahrens mit gewöhnlichen Matrixverfahren. *Math. Ann.* 131 (1956), 253-257.

Let  $B = (b_{nm})$  denote a matrix method for summing infinite series, and let  $A$  denote the Abel method. Earlier, the author showed that if  $B$  is row-finite, then its convergence field can not be identical with that of  $A$  [*C. R. Acad. Sci. Paris* 236 (1953), 568-569; MR 14, 744]. He now extends this result to all matrix methods  $B$ . In the proof, the convergence fields of  $A$  and  $B$  are regarded as an  $F$ -space and an  $FK$ -space, respectively. The main argument runs as follows: The Banach space of the continuous functions onto which  $A$  maps its convergence field has a nonseparable conjugate; the corresponding Banach space of convergent sequences associated with  $B$  does not have this property.

The proof makes use of the following lemma, which is of independent interest: If the convergence field of  $B$  includes that of  $A$ , then there exist numbers  $r$  and  $R$  ( $r > 1$ ) such that

$$b_{nm} = \int_0^1 y^m dg_n(y) + c_{nm},$$

where the variation of  $g_n(y)$  is less than  $R$ , and where  $\sum_{m=0}^{\infty} |c_{nm}| r^m < R$ . This implies the additional result that if  $B$  is a regular row-finite matrix method, then its convergence field does not contain the convergence field of  $A$ .

G. Piranian (Ann Arbor, Mich.).

El Makarem, H. H. A. Some results on matrix spaces. I, II. *Nederl. Akad. Wetensch. Proc. Ser. A.* 59=Indag. Math. 18 (1956), 490-498, 499-510.

All the results given in this paper have been previously

published by the author [Some results on matrix spaces, Review of Economics, Political, Business studies, Faculty of Commerce, Cairo University 4 (1956), no. 1]. They consist of necessary and sufficient conditions for a matrix to transform one sequence space of specified type into another, with allied results on projective boundedness, projective convergence, and strong projective convergence of a sequence of matrices. (The reviewer had already conveyed to the author, before the latter published these papers, that all these isolated results, with their similar proofs, are very probably particular cases of a small number of much more general theorems. As an example of this, the author's (II), p. 493, is a particular case of a theorem recently given by Allen [J. London Math. Soc. 31 (1956), 374-376, Th. I, p. 374; MR 18, 31]. The author's condition (iii) in (II), p. 493, can in any case be replaced by the simpler condition that  $\sum_{n=1}^{\infty} |\sum_{k=1}^{\infty} a_{n,k}|$  converges.

If the author had succeeded in covering all his results by a few theorems of greater generality, the results would have been more impressive. R. G. Cooke (London).

**Bondar', V. P.** A class of convergence criteria. Grodnenskiĭ Gos. Ped. Inst. Uč. Zap. 1 (1955), 17-20. (Russian)

**Streleckiĭ, È. V.** Example of a series summable (A) and not summable (C,  $\rho$ ) ( $\rho > 0$ ). Grodnenskiĭ Gos. Ped. Inst. Uč. Zap. 1 (1955), 71-72. (Russian)

**Streleckiĭ, È. V.** Chain of convergence criteria for series with positive terms. Grodnenskiĭ Gos. Ped. Inst. Uč. Zap. 1 (1955), 67-69. (Russian)

**Schmeidler, Werner.** Über symmetrische algebraische Integralgleichungen. Ann. Acad. Sci. Fenn. Ser. A. I. no. 220 (1956), 18 pp.

Let  $n$  be a positive odd integer, and for  $\beta = 0, 1, \dots, n-1$ ,  $\nu = (n+1)/(\beta+1)-1$  let  $K_{\beta}(s, t_1, \dots, t_{\nu})$  be continuous real valued symmetric functions with the unit cube as domain. The author considers the "eigenvalue problem" for the symmetric algebraic integral equation

$$(1) \varphi[y] = \left\{ \mu^n y^n(s) - \sum_{\beta=0}^{n-1} \mu^{\beta} y^{\beta}(s) \int_0^1 \dots \int_0^1 K_{\beta}(s, t_1, \dots, t_{\nu}) \times y^{\beta+1}(t_1) \dots y^{\beta+1}(t_{\nu}) dt_1 \dots dt_{\nu} = 0 \right\}.$$

It is proved that the following condition is sufficient for the existence of a real constant  $\mu$  and a continuous  $y(s)$  satisfying (1): if for fixed continuous  $y=y(s)$ ,  $z_0[y]$  denotes the absolutely greatest root (which may be supposed to be positive) of the algebraic equation  $\varphi[y]=0$  for  $\mu$ , and if  $\mu_0$  is a positive constant then there exists a positive constant  $\gamma=\gamma(\mu_0)$  such that the discriminant of  $\varphi[y]$  (as polynomial in  $\mu$ ) is  $>\gamma(\mu_0)$  for all  $y$  with  $\|y\|=1$  for which  $z_0(y) \geq \mu_0$ . Here the norm  $\|y\|$  is defined by

$$\|y\|^{n+1} = \int_0^1 y^{n+1}(s) ds.$$

In generalization of a procedure which is classical in the case of a linear symmetric integral equation ( $n=1$ ), the eigenvalue is constructed as the maximum of  $z_0[y]$  for all continuous  $y$  with  $\|y\|=1$ , and the corresponding eigenfunction is such a  $y$  maximizing  $z_0[y]$ .

Finally, a sufficient condition is given for (1) to have at most a countable number of eigenvalues. E. H. Rothe.

See also: Nalli, p. 274; Abhyankar, p. 277; Schwartz, p. 287; Barocio, p. 289; Parodi, p. 300; Hull, p. 302; Kalafati, p. 310; Montaldo, p. 314; Sobolev, p. 322; Flemming, p. 336; Thullen, p. 340; Dalcher, p. 343; Rozovskil', p. 350; Baratta, p. 358; Cuénad, p. 365.

### Orthogonal Functions, Approximations

**Moldovan, Elena.** Sur certains théorèmes de moyenne. Com. Acad. R. P. Roum. 6 (1956), 7-12. (Romanian. Russian and French summaries)

Let  $\mathcal{F}$  be a family of functions  $F(x; a_1, \dots, a_n)$  satisfying (i)  $F$  is continuous for  $a \leq x \leq b$ ,  $\alpha_i \leq a_i \leq \beta_i$  ( $i=1, \dots, n$ ) and (ii) for  $n$  arbitrary points  $x_1, \dots, x_n$  in  $[a, b]$  and arbitrary  $y_i$  the interpolating equations

$$F(x_i; a_1, \dots, a_n) = y_i \quad (i=1, \dots, n)$$

considered as a system of equations for  $a_1, \dots, a_n$  have a unique solution in  $\mathcal{F}$ . The function obtained in (ii) is denoted by  $L(x_1, \dots, x_n; y|x)$ . Let  $a \leq x_1 < x_2 < \dots < x_{n+1} \leq b$  and denote  $L(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}; y|x)$  by  $L_i(x)$ . One of the principal theorems of the paper asserts that for any  $x_0$ ,  $x_{n+1} < x_0 < b$ ,  $L_i(x_0)$  ( $i=2, \dots, n$ ) is always between  $L_1(x_0)$  and  $L_{n+1}(x_0)$ , and there are corresponding results for other positions of  $x_0$ , and for larger numbers of points.

A. Erdélyi (Jerusalem).

**Hull, T. E.** Some algebraic properties of asymptotic power series. Canad. J. Math. 8 (1956), 220-224.

The collection of all asymptotic series from a ring under formal addition, subtraction, and multiplication. The ring is isomorphic to the ring of all asymptotic sums. A special case of a theorem of Borel is: To any series  $\sum_{m=0}^{\infty} c_m x^m$  there corresponds at least one function  $f(x)$  such that

$$R_n(x) \cdot x^n = f(x) - \sum_{m=0}^{n-1} c_m x^m = o(x^{n-1}), \quad x \rightarrow 0,$$

$$(n=1, 2, \dots; c_m \text{ real, } x \text{ real, non-negative}).$$

Particular attention is paid to those special asymptotic series for which (i)  $c_0 \geq 0$ ; (ii) there exists a sum function  $f(x)$  such that, for all  $x > 0$

$$|R_n(x)| \leq |c_n| \quad (n=0, 1, 2, \dots; R_0(x) = f(x)).$$

Any series satisfying (i) and (ii) is called an S-series. The author shows: Th. I. The S-series form a semiring that is, the formal sum or product of two S-series is an S-series and the distributive law holds. Th. II. Any asymptotic series can be written as the difference between two S-series. Incidentally, it is shown that the full ring of all asymptotic series is generated from the semiring of all S-series, when all differences are adjoined to the semiring.

S. C. van Veen (Delft).

See also: Hayman, p. 293; Allen, p. 300; Steĭkin, p. 303; Rutovitz, p. 309; Kalafati, p. 310; Kac, p. 310; Levitan, p. 312; Zygmund, p. 321; Sobolev, p. 322; Salzer, p. 339.

**Trigonometric Series and Integrals**

**Stečkin, S. B.** On best approximation of certain classes of periodic functions by trigonometric polynomials. *Izv. Akad. Nauk SSSR. Ser. Mat.* 20 (1956), 643-648. (Russian)

Let  $r$  be positive and  $\alpha$  real, and put

$$\Psi_r(x, \alpha) = \sum_{k=1}^{\infty} k^{-r} \cos(kx - \alpha\pi/2).$$

Then let  $W^{(r)}(\alpha)$  denote the class of continuous periodic functions  $f$  such that

$$f(x) = \frac{1}{2}a_0 + \pi^{-1} \int_0^{2\pi} \Psi_r(x-t, \alpha) \varphi(t) dt$$

with  $\int_0^{2\pi} \varphi(t) dt = 0$  and  $|\varphi(t)| \leq 1$  almost everywhere; let  $W_1^{(r)}(\alpha)$  be defined similarly with  $\int_0^{2\pi} |\varphi(t)| dt \leq 1$ . Then, for example,  $W^{(r)}(\alpha)$  is the class  $W^{(r)}$  of continuous functions with (Weyl) derivative  $f^{(r)}$  bounded almost everywhere by 1, and  $W^{(r)}(\alpha+1)$  is the corresponding class with  $f^{(r)}$  bounded almost everywhere by 1. Let  $E_n[H]$  denote the best approximation to  $f$  in the class  $H$  by trigonometric polynomials of order  $n-1$ , and  $E_n[H]_L$  the same thing in the  $L$  metric. For  $H = W^{(r)}(0)$ ,  $W^{(r)}(1)$ ;  $W_1^{(r)}(0)$ ,  $W_1^{(r)}(1)$ ;  $W^{(r)}(\alpha)$ ,  $W_1^{(r)}(\alpha)$ , the best appropriate approximations have been determined by Sz.-Nagy [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 90 (1938), 103-134], Nikolsky [Izv. Akad. Nauk SSSR. Ser. Mat. 10 (1946), 207-256; MR 8, 149] and Dzyadyk [ibid. 17 (1953), 135-162; MR 14, 867]. The author now shows that for  $0 < r < 1$ ,  $r \leq \alpha \leq 2-r$ ,

$$E_n[W^{(r)}(\alpha)] = E_n[W_1^{(r)}(\alpha)]_L = 4\pi^{-1} K_r n^{-r} \sin \frac{1}{2}\alpha\pi,$$

where  $K_r = \sum_{j=0}^{\infty} (2j+1)^{-r-1}$ . For  $0 < r \leq \frac{1}{2}$  this implies results on the classes  $\tilde{W}^{(r)}$  and  $\tilde{W}_1^{(r)}$ .

R. P. Boas, Jr. (Evanston, Ill.).

**Chen, Yung-Ming.** On the integrability of functions defined by trigonometrical series. *Math. Z.* 66 (1956), 9-12.

Let  $f(x)$  be defined by a cosine or sine series with coefficients  $\lambda_n$  decreasing monotonically to zero. Then if  $p > 1$  and  $0 < Y < 1$ , we have  $x^{-Y}\{f(x)\}^p \in L(0, \pi)$  if and only if  $\sum n^{Y+p-2} \lambda_n^p$  converges. If  $\{\lambda_n\}$  is also convex,  $n\lambda_n$  is nondecreasing, and  $f$  is defined by a sine series, the same conclusion holds for every positive  $Y$ . R. P. Boas, Jr.

**Arsac, J.** Application des théories de l'approximation à l'étude des images optiques. *Opt. Acta* 3 (1956), 55-65.

A theorem of Bernstein gives an upper bound for the absolute value of the derivative of a finite Fourier series. In the present paper an extension of this theorem is given and is applied to some questions relating to the imagery of an incoherent object by an optical system that is assumed to behave as a linear filter. [The extended theorem was also given by Bernstein; cf. Boas, Entire Functions, New York, 1954, pp. 206 ff.] E. Wolf (New York, N.Y.).

**Gosselin, Richard P.** On the convergence of Fourier series of functions in an  $L^p$  class. *Proc. Amer. Math. Soc.* 7 (1956), 392-397.

Let  $f(x)$  be of class  $L^p(0, 2\pi)$ , and let  $S_n(x; f)$  denote the partial sums of the Fourier expansion of  $f(x)$ . A sequence  $\{n_k\}$  is called lacunary if  $n_{k+1}/n_k \geq \lambda > 1$ . If  $f(x) \in L^2(0, 2\pi)$ , then according to a well-known theorem of Kolmogoroff,  $S_{n_k}(x; f)$  converges to  $f(x)$  almost everywhere. Let  $L_k = [(n_{k+1} - n_k)/\log n_{k+1}]$ , where  $[y]$  denotes the greatest

integer in  $y$ . The author proves that if  $f(x) \in L^2(0, 2\pi)$ , and  $\{n_k\}$  is lacunary, then there exists a sequence of positive integers  $\{m_r\}$  containing  $L_k$  consecutive terms in each interval  $(n_k, n_{k+1})$  such that the subsequence  $S_{m_r}(x; f)$  converges to  $f(x)$  almost everywhere.

While the subsequence obtained by the author is denser than that of Kolmogoroff's theorem, its location is less precise. A similar, but more complicated result is obtained for  $f(x) \in L^p$  ( $1 < p < 2$ ). The author also proves the following: let  $f(x) \in L^2(0, 2\pi)$ , then there exists a subsequence  $\{m_r\}$  of upper density one, such that the subsequence  $S_{m_r}(x; f)$  converges to  $f(x)$  almost everywhere.

The above theorems are proved by breaking up the Fourier development of  $f(x)$  into appropriate blocks, and then applying the following well-known theorem of Kolmogoroff-Seliverstoff and Plessner: If  $\sum |a_n|^2 \log |n|$  converges, then the Fourier series of  $\sum a_n e^{inx}$  converges almost everywhere. E. M. Stein.

**Bari, N. K.; and Stečkin, S. B.** Best approximations and differential properties of two conjugate functions. *Trudy Moskov. Mat. Obšč.* 5 (1956), 483-522. (Russian)

Let  $f(x)$  be a continuous function of period  $2\pi$  and  $\tilde{f}(x)$  its conjugate. The paper investigates relations between various differential properties of  $f$  and  $\tilde{f}$  and the best approximations  $E_n(f)$ ,  $E_n(\tilde{f})$  of  $f$  and  $\tilde{f}$  by trigonometric polynomials of order  $\leq n$ . It has points of contact with previous papers of the authors [see Bari, Izv. Akad. Nauk SSSR. Ser. Mat. 19 (1955), 285-302; MR 17, 256; Stečkin, ibid. 20 (1956), 197-206; MR 17, 1079]. Given any integer  $k \geq 1$ , let  $\omega_k(\delta, f)$  be the modulus of continuity of  $f$  of order  $k$ :

$$\omega_k(\delta, f) = \max_{|h| \leq \delta} \max_x \left| \sum_{j=0}^k (-1)^j C_k^j f(x+jh) \right|;$$

we write  $\omega$  for  $\omega_1$ . A function  $\varphi(t)$ ,  $0 \leq \delta \leq \pi$ , is said to belong to class  $\Phi$ , if  $\varphi$  is continuous, non-decreasing, 0 at the origin and positive elsewhere. A  $\varphi \in \Phi$  is said to belong to class  $B$ , if  $\sum_{n=1}^{\infty} n^{-1} \varphi(n^{-1}) = O\{\varphi(1/n)\}$ , and to class  $B_k$  ( $k=1, 2, \dots$ ) if  $\sum_{n=1}^{\infty} n^{k-1} \varphi(n^{-1}) = O\{n^k \varphi(1/n)\}$ . The following are the main results of the paper. 1) If  $\varphi \in B$ ,  $r$  is a non-negative integer, then all the relations

$$\begin{aligned} (*) & E_n(f^{(r)}) = O\{n^{-(r-p)} \varphi(1/n)\}, \\ (**) & E_n(\tilde{f}^{(r)}) = O\{n^{-(r-q)} \varphi(1/n)\}, \end{aligned}$$

where  $p, q=0, 1, \dots, r$ , are equivalent. 2) Condition  $B$  is necessary for the validity of (\*) and (\*\*) for all  $f$ . 3) Let  $k$  be positive,  $r$  non-negative, both integers, and suppose that  $\varphi \in B$ ,  $\varphi \in B_k$ . Then the four relations (\*), (\*\*), and

$$(\dagger) \omega_{k+r-s}(\delta, f^{(s)}) = O\{\delta^{r-s} \varphi(\delta)\}, \omega_{k+r-m}(\delta, \tilde{f}^{(m)}) = O\{\delta^{r-m} \varphi(\delta)\}$$

where  $p, q, r, s=0, 1, \dots, r$ , are all equivalent. 4) The preceding propositions hold if the relations ' $=O$ ' are replaced by ' $\sim$ ' (we write  $A_n \sim B_n$  if the ratios  $A_n/B_n$  and  $B_n/A_n$  are bounded). 5) Suppose that  $\varphi \in B$  and  $\varphi(\delta)/\delta$  decreases; then the relations  $\omega(\delta, f) = O\{\varphi(\delta)\}$  and  $\omega(\delta, \tilde{f}) = O\{\varphi(\delta)\}$  are equivalent if and only if  $\varphi$  satisfies both conditions  $B$  and  $B_1$ . A. Zygmund.

**Fedulov, V. S.** Generalization of two theorems of A. N. Kolmogorov or lacunary sequences. *Uč. Zap. Kazan. Univ.* 115 (1955), no. 14, 87-95. (Russian)

Let (1)  $f(x, y)$  be periodic ( $2\pi$ ) and Lebesgue integrable over the square  $Q: |x| \leq \pi, |y| \leq \pi$ ; and let  $S_{nm}(x, y)$  denote the  $(n, m)$ th rectangular partial sum of the double Fourier series of  $f(x, y)$ . A sequence of indices  $n_k$  is called lacunary

if  $\inf n_{k+1}/n_k > 1$ ; a sequence of index pairs  $(n_k, m_k)$  is lacunary if both  $\{n_k\}$  and  $\{m_k\}$  are lacunary; and a double trigonometric series is lacunary if its coefficients are zero, except for a set of coefficients whose first and second indices belong to lacunary sequences  $\{n_k\}$  and  $\{m_k\}$ , respectively. Two theorems of Kolmogoroff [Fund. Math. 5 (1924), 96-97] are extended to functions of two variables: If (1) is in  $L^2$  on  $Q$  and if the sequence  $\{n_k, m_k\}$  is lacunary, then  $S_{n_k, m_k}(x, y) \rightarrow f(x, y)$  almost everywhere in  $Q$ . If  $f \in L$  on  $Q$  and if its Fourier series is lacunary, then  $S_{nm}(x, y) \rightarrow f(x, y)$  almost everywhere in  $Q$ , as  $n \rightarrow \infty$  and  $m \rightarrow \infty$ .

G. Piranian (Ann Arbor, Mich.).

**Mihlin, S. G. On the multipliers of Fourier integrals.**

Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 701-703. (Russian)

Let  $g(x) = g(x_1, \dots, x_m)$  be of period  $2\pi$  in each  $x_j$ , integrable, and let  $g(x) \sim \sum a_n e^{i(n, x)}$ , where  $(n, x) = n_1 x_1 + \dots + n_m x_m$  and  $n = (n_1, \dots, n_m)$  is a lattice point with integral coordinates. Consider also

$$h(x) \sim \sum a_n \lambda(n) e^{i(n, x)}.$$

Marcinkiewicz [Studia Math. 8 (1939), 78-91] gave a sufficient condition for the multipliers  $\lambda(n)$  to transform  $L^p$  into  $L^p$ ,  $1 < p < \infty$ , and showed that then

$$\|h\|_p \leq A_{p,m} \lambda \|g\|_p.$$

The condition is not simple enough to be restated here; In the present paper, the author starting from the result it contains differences of order  $m$  of  $\lambda(n) = \lambda(n_1, \dots, n_m)$  of Marcinkiewicz obtains, through a passage to the limit, the following result. Let  $E^m$  be the Euclidean  $m$ -space, and suppose that, except possibly at the origin,  $\Phi(x)$  is continuous, has the derivative  $\partial^m \Phi / \partial x_1 \dots \partial x_m$ , and that all derivatives of order  $< m$  exist and are continuous. Suppose also that  $|x|^k |D^k \Phi| \leq M$ , where  $D^k \Phi$  is any of the derivatives mentioned above. Then the operator

$$F(g) = h(x) = (2\pi)^{-m/2} \int_{E^m} e^{i(x, y)} \Phi(y) dy \int_{E^m} e^{-i(y, z)} g(z) dz$$

is defined on a set dense in  $L^p(E^m)$ ,  $1 < p < \infty$ , and is bounded. The result is applied to the singular operator

$$\int_{E^m} f(\theta) r^{-m} \gamma(y) dy,$$

where  $r = |x - y|$  and  $\theta$  is the projection of  $y$  onto the unit sphere with center at  $O$ .

A. Zygmund.

**Goldberg, Richard R.; and Varga, Richard S. Möbius inversion of Fourier transforms.** Duke Math. J. 23 (1956), 553-559.

Let

$$F(t) = \int_0^\infty \phi(u) \cos tu \, du, \quad \phi(t) = (2/\pi) \int_0^\infty F(u) \cos tu \, du.$$

Let  $\mu(n)$  be the Möbius function of number theory. Then if  $\phi(u)$  is of bounded variation in every finite interval and  $\phi(u) \log u \in L(1, \infty)$ , the authors show that

$$G(t) = t^{-1} \left\{ \frac{1}{2} F(0) + \sum_1^\infty (-1)^k F(k\pi/t) \right\}$$

is finite almost everywhere and

$$\phi(t) = \sum_{n=1}^\infty \mu(2n-1) G\{(2n-1)t\}$$

almost everywhere. Under mild additional hypotheses,

for example  $\phi(t) = O(t^{-\alpha})$ ,  $\alpha = 1$ , the conclusions hold everywhere. The proof depends on the Möbius inversion formula, two lemmas on sums, and a form of the Poisson summation formula. The latter are as follows. (1) If  $\phi(t) \in L(R, \infty)$  for every positive  $R$  then  $\sum |\phi(kt)|$  converges almost everywhere. (2) If also  $\phi(t) \log t \in L(1, \infty)$ , then  $\sum \sum |\phi(knt)|$  converges almost everywhere. (3) If  $\phi$  is of bounded variation on every  $(0, R)$  and  $\phi \in L(R, \infty)$  for some  $R$ , then  $G(t)$  (defined above) exists almost everywhere and  $G(t) = \sum \phi\{(2k-1)t\}$  almost everywhere. [The functions of bounded variation are supposed to be normalised, although this condition was inadvertently omitted in the paper.] R. P. Boas, Jr. (Evanston, Ill.).

See also: Teghem, p. 286; Schwartz, p. 288; Mandelbrojt, p. 305; Levitan and Sargeyan, p. 310; Lipschutz, p. 340; Kovalenko, p. 349; Prem, p. 352; Kovtun, p. 359.

### Integral Transforms

**★ Widder, D. V. The heat equation and the Weierstrass transform.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 227-234. University of Maryland Book Store, College Park, Md., 1956.

This is an exposition of properties of the Weierstrass transform, whose kernel is  $(4\pi)^{-1/2} e^{-(x-y)^2/4}$ , and its relation to the heat equation. [A more detailed account is given in Hirschman and Widder, "The convolution transform" (Princeton, 1955; MR 17, 479).] R. P. Boas, Jr.

**Akutowicz, Edwin J. On the determination of the phase of a Fourier integral. I.** Trans. Amer. Math. Soc. 83 (1956), 179-192.

Let  $f$  belong to  $L$  and  $L^2$  on the real axis and let  $F$  be its Fourier transform. The author seeks to find the extent to which  $|F|$  determines  $f$ , and finds that even under fairly strong additional conditions there is more indeterminateness than physicists would like. Let  $a(x)$  be a given function not vanishing for real  $x$ ; then if  $f_1(x) = f_2(x) = 0$  almost everywhere for  $x < 0$  and  $|F_1(x)| = |F_2(x)| = a(x)$  for all real  $x$ , the author proves that

$$e^{ic_1 + ib_1 x} B_1(x) F_1(x) = e^{ic_2 + ib_2 x} B_2(x) F_2(x),$$

where the  $c$ 's are real, the  $b$ 's are nonnegative, and the  $B$ 's are analytic in the closed upper half-plane, of modulus 1 on the real axis, and are in fact given by Blaschke products. He also shows that if there is a solution  $f$  for a given  $a$  such that  $F(x)$  has sufficiently few zeros in the upper half-plane, then there is also a solution such that  $F$  has no zeros in the upper half-plane. An example shows that even if we make the additional assumption that  $f(x) \geq 0$  then  $|F|$  still does not determine  $f$  up to trivial factors (that is, the Blaschke products cannot be excluded). Since the methods of the paper are inherently one-dimensional, the crystallographers' problem of whether  $|F|$  determines  $f$ , assumed non-negative, remains open in the three-dimensional case.

R. P. Boas, Jr.

**Koosis, Paul. Note sur les fonctions moyenne-périodiques.** Ann. Inst. Fourier, Grenoble 6 (1955-1956), 357-360.

Nouvelle démonstration du théorème fondamental sur les fonctions moyenne-périodiques: si la transformée de Fourier-Carleman de  $f$  n'a pas de pôles,  $f$  se réduit à zéro.

J. P. Kahane (Montpellier).

**Mandelbrojt, S.** La transformée de Fourier et les fonctions holomorphes dans un demi-plan. *J. Math. Pures Appl.* (9) 35 (1956), 211-222.

If  $F \in L(-\infty, \infty)$ ,  $f$  denotes its Fourier transform,  $f(u) = (2\pi)^{-1} \int_{-\infty}^{\infty} F(x) e^{-iux} dx$ . The author denotes by  $T$  the subset of  $L$  consisting of continuous functions for which  $\sum \max_{n, n+1} |F(x)| < \infty$ ; by  $B$  the set of essentially bounded measurable functions; by  $W$ , the set of functions  $F$  which are locally of bounded variation and have  $\int_{-\infty}^{\infty} |dF|$  bounded. Theorem 1. If  $K \in L$  and  $h(u) \neq 0$  in  $(-\infty, -h)$ ; if  $F \in B$  and  $M$  is the essential maximum of  $|F|$ ; and if  $\int K(y-x)F(x)dx=0$  then there is an  $F_0(z)$  regular in  $y>0$  such that  $|F_0(z)| \leq M e^{h y}$  and

$$(*) \quad \lim_{y \rightarrow 0+} \int_{-N}^N |F_0(z) - F(x)| dx = 0$$

for every positive  $N$ . Theorem 2. If  $K \in T$  and  $h \neq 0$  in  $(-\infty, -h)$ ,  $h \geq 0$ ; if  $F \in W$  and  $\int K(y-x)dF(x)=0$ , then there is an  $F_0(z)$ , regular in  $y>0$ , satisfying (\*) and

$$|x^{-1}(e^{tez}-1)F_0(z)| \leq \sup_{-\infty < z < \infty} |x^{-1}(e^{tez}-1)F(x)| e^{h y}.$$

If  $h \neq 0$  both in a left-hand and a right-hand interval, there are similar conclusions in which  $F_0(z)$  is entire.

R. P. Boas, Jr. (Evanston, Ill.).

**Tanno, Yûkichi.** An inversion formula for convolution transforms. *Kôdai Math. Sem. Rep.* 8 (1956), 79-84.

The author gives a detailed exposition of a result merely sketched by the reviewer and Widder [*Duke Math. J.* 15 (1948), 659-696; MR 10, 371].

I. I. Hirschman, Jr. (St. Louis, Mo.).

See also: Schwartz, p. 288; Krumbach, p. 288; Parodi, p. 300; Hatcher, p. 300; Goldberg and Varga, p. 304; Crum, p. 309; Flemming, p. 336; Hurwitz and Zweifel, p. 337; Goodwin, p. 338; Maruyama, p. 341; Dalcher, p. 343; Radok, p. 349; Papadopoulos, p. 357; Parkus, p. 354; Bracewell, p. 365.

### Ordinary Differential Equations

★ **Haas, Felix.** On the total number of singular points and limit cycles of a differential equation. Contributions to the theory of nonlinear oscillations, vol. 3, pp. 137-172. *Annals of Mathematics Studies*, no. 36. Princeton University Press, Princeton, N. J., 1956. \$4.00.

Inequalities between the number of singular points and limit cycles of a differential equation defined as a vector field on a manifold and the Betti numbers of the manifold are secured. Each manifold considered is a union of disjoint, twice-differentiable, two-dimensional, closed manifolds. A theorem of El'sgol'c [*Mat. Sb. N.S.* 26(68) (1950), 215-223; MR 11, 671; 13, 850] asserting that the zeroth, first and second Betti numbers of the manifold are less than or equal to, respectively, the number of unstable singular points, the number of saddle points and the number of stable singular points of the differential equation is proved in detail. In this theorem the differential equation is to have only a finite number of simple singularities and through each ordinary point there is to be a unique trajectory. In the remainder of the paper the differential equation may have complicated singularities and, in some cases, uniqueness of trajectories is relaxed. The inequalities established for such general differential equations are difficult to describe in a review.

W. R. Utz (Columbia, Mo.).

**Ilinskii, A. Yu.** Approximate method of investigation of vibrational systems. *Akad. Nauk Ukrain. RSR. Prikl. Meh.* 2 (1956), 152-158. (Ukrainian. Russian summary)

The method consists in reducing an ordinary differential equation (A) in  $x$  to an equation (A) of the second order by substituting  $x = a' \cos(pt + \alpha)$  where  $a'$ ,  $p$ ,  $\alpha$  are independent of  $t$ , and then assuming that  $p$  is the circular frequency of the free undamped vibrations described by (A). The following example should be sufficient to show the scope of the method. If the original equation is homogeneous of the third order, (A) becomes  $\ddot{x} + (b - ap^2)\dot{x} + cx = 0$ , and stability requires  $b > ap^2$ . Substituting  $c$  for  $p^2$ , the correct Routh-Hurwitz stability condition,  $b > ca$ , is obtained for the original equation. If, however, (A) is to be satisfied by the trial solution,  $p^2 = \frac{1}{2}(c - (b - p^2))$  and  $p^2 = 2c - b > 0$ , a condition weaker than the true one, unless supplemented by  $c > p^2$ . The remaining condition,  $a > 0$ , cannot be derived by this method.

A. W. Wundheiler (Chicago, Ill.).

★ **Seifert, George.** A rotated vector approach to the problem of stability of solutions of pendulum-type equations. Contributions to the theory of nonlinear oscillations, vol. 3, pp. 1-16. *Annals of Mathematics Studies*, no. 36. Princeton University Press, Princeton, N. J., 1956. \$4.00.

Consider the differential equation

$$(1) \quad \theta'' + f(\theta, \alpha)\theta' = g(\theta),$$

where  $' = d/dt$ ,  $f(\theta, \alpha) \in C^2$ ,  $\alpha > 0$ ,  $-\infty < \theta < +\infty$ ,  $f(\theta, \alpha) > 0$ ,  $\partial f/\partial \alpha > 0$ ,  $f(\theta + 2\pi, \alpha) = f(\theta, \alpha)$ ;  $g(\theta) \in C^2$ ,  $-\infty < \theta < +\infty$ ,  $g(\theta + 2\pi) = g(\theta)$ ,  $g(\theta)$  has zeros  $\theta_i$  such that  $g'(\theta_i) \neq 0$  and  $\int_{\theta_i}^{\theta_i+2\pi} g(\theta)d\theta > 0$ . If (1) has a solution  $\theta(t, \alpha)$  such that  $\theta'(t, \alpha) = y(\theta, \alpha)$  where  $y(\theta + 2\pi, \alpha) = y(\theta, \alpha)$ , then  $\theta(t, \alpha)$  is a periodic solution of the second kind of (1). It is known [Amerio, *Ann. Scuola Norm. Sup. Pisa* (3) 3 (1949), 19-57; MR 12, 180] that for fixed  $\alpha$  if such a solution exists, it is unique and its time derivative is non-negative, while if no such solution exists, then each solution of (1) tends to a finite limit as  $t \rightarrow \infty$ , i.e. is stable. Let  $L^*$  be the set of  $\alpha$  for which periodic solutions of the second kind of (1) exist; let  $P_2^*$  be the class of functions  $y(\theta, \alpha) = \theta'(t, \alpha)$ ,  $\alpha \in L^*$ ; let  $P_2$  be the subset of  $P_2^*$  consisting of the positive functions of  $P_2^*$ ; and let  $L$  be the subset of  $L^*$  for which  $y(\theta, \alpha) \in P_2$ . The author proves that, if  $P_2$  is not empty,  $L^*$  is an interval (not necessarily finite);  $L$  is an open interval with lower limit zero;  $y(\theta, \alpha) \in P_2$  is a strictly decreasing function of  $\alpha$  for  $\alpha \in L$ ; and  $L$  bounded implies  $L^*$  consists of  $L$  and the least upper bound of  $L$  and  $P_2^*$  consist of  $P_2$  plus a periodic solution  $y(\theta, \alpha) = \theta'(t, \alpha)$  which has a zero. If  $f(\theta, \alpha) = \alpha f(\theta)$ , then the author has shown previously [*Quart. Appl. Math.* 11 (1953), 127-131; MR 15, 127] that  $P_2$  is not empty. He shows in this paper that in this case there is a finite upper bound for  $L$  and gives a method for the estimation of this upper bound.

J. K. Hale (Albuquerque, N.M.).

**Levin, J. J.** Singular perturbations of nonlinear systems of differential equations related to conditional stability. *Duke Math. J.* 23 (1956), 609-620.

Consider the system of differential equations

$$(1) \quad x' = f(t, x, y, \epsilon), \quad \epsilon y' = g(t, x, y, \epsilon)$$

and the related system (2)  $x' = f(t, x, y, 0)$ ,  $0 = g(t, x, y, 0)$  and let  $f_x$  be the  $m \times n$  matrix whose components are  $\partial f_i / \partial x_j$ . Suppose  $x = p(t)$ ,  $y = q(t)$  is a solution of (2) for

$0 \leq t \leq T$ ;  $f, g, f_x, f_y, g_x, g_y$  are in  $C$  if  $|x - p(t)| + |y - q(t)| + \varepsilon$  is sufficiently small for  $0 \leq t \leq T$ ; and there exists a non-singular real matrix  $P(t) \in C^1$  ( $0 \leq t \leq T$ ) such that

$$P^{-1}(t)g_y(t)P(t) = \text{diag}(B(t), C(t)),$$

where  $g_y(t) = g_y(t, p(t), q(t), 0)$ ,  $B(t)$  is an  $n_1 \times n_1$  matrix whose characteristic roots have real part negative,  $0 \leq t \leq T$ , and  $C(t)$  is an  $n_2 \times n_2$  matrix whose characteristic roots have real part positive,  $0 \leq t \leq T$ ,  $n_1 + n_2 = n$ . Systems (1), (2) were discussed by the author and N. Levinson [J. Rational Mech. Anal. 3 (1954), 247-270; MR 15, 795] for the case  $n_2 = 0$ . Systems (1), (2) were discussed by L. Flatto and N. Levinson [ibid. 4 (1955), 943-950; MR 17, 849] for  $n_2 \neq 0$ ,  $p(t)$ ,  $q(t)$ ,  $f$ ,  $g$  periodic in  $t$  of period  $T$ . The purpose of the present paper is to discuss (1), (2) for  $n_2 \neq 0$  and  $p, q, f, g$  not necessarily periodic. The author shows there exists a constant  $\gamma_1 > 0$ , independent of  $\varepsilon$ , and an  $n_1$ -dimensional manifold,  $S(a, \varepsilon)$ , in  $y$  space which depends continuously on  $(a, \varepsilon)$  for  $|a| \leq \gamma_1$ ,  $0 < \varepsilon \leq \gamma_1$ , such that  $x(0, \varepsilon) = p(0) + a$ ,  $y(0, \varepsilon) \in S(a, \varepsilon)$  implies the solution  $x(t, \varepsilon)$ ,  $y(t, \varepsilon)$  of (1) is unique, exists over  $0 \leq t \leq T$  for all  $\varepsilon$ ,  $0 < \varepsilon \leq \gamma_1$  and approaches the solution  $p, q$  of (2) as  $|a|, \varepsilon \rightarrow 0$ . It is also shown that  $y(0, \varepsilon)$  need not lie precisely on  $S(a, \varepsilon)$  for the above result to hold. Furthermore, if a solution of (1) is sufficiently close to  $p(t), q(t)$  over  $0 \leq t \leq T$  for  $0 < \varepsilon \leq \gamma_1$ , then  $y(0, \varepsilon) \rightarrow S(a, \varepsilon)$  exponentially fast as  $\varepsilon \rightarrow 0$ . J. K. Hale (Albuquerque, N.M.).

**Yataev, M.** On a countable system of differential equations in  $L_p$ . Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. 1956, no. 4(8), 12-23. (Russian)

Let  $x(p)$  be the point with coordinates  $x_1(p), x_2(p), \dots$ , whose space is  $L_p$  normed by

$$\|x\| = \sup \left[ \int_{\alpha}^{\beta} |x_s(p)|^p dp \right]^{1/p},$$

where  $[\alpha, \beta]$  is fixed. The author studies solutions of a system

$$\dot{x} = \omega(t; x) \quad (t \geq 0),$$

where  $x \in L_p$ , and the initial value is  $x^0(p)$ . Under rather general assumptions of more or less standard type he proves the existence and uniqueness of the solution as well as the stability theorems of Lyapunov.

S. Lefschetz (Princeton, N.J.).

**Karaseva, T. M.** On a criterion of boundedness of the solutions of Hill's differential equations. Prikl. Mat. Meh. 20 (1956), 549-551. (Russian)

Krein [Prikl. Mat. Meh. 19 (1955), 641-680; MR 17, 1088] proved that all solutions of  $\dot{x} + p(t)x = 0$  with  $p(t+T) = p(t)$  and  $\int_0^T p(t)dt = 0$  are bounded if

$$T \int_0^T q^2(t)dt < \pi^2/4,$$

where  $\dot{q}(t) = p(t)$ ,  $\int_0^T q(t)dt = 0$ ; he conjectured that this criterion would remain valid with  $\pi^2/4$  replaced by 3. The author shows that  $\pi^2/4$  is best possible.

H. A. Antosiewicz (Washington, D.C.).

**Tatarkiewicz, Krzysztof.** Propriétés asymptotiques des systèmes d'équations différentielles ordinaires presque linéaires. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 8 (1954), 25-69 (1956). (Polish and Russian summaries)

Consider in  $E^n$ : (\*)  $\dot{x} = Ax + h_1(t, x) + h_2(t, x) + f(t)$ , where  $A$  is a constant matrix whose eigenvalues have  $p$  distinct

real parts,  $\rho_1 < \dots < \rho_p$ ,  $f \in C$  on  $L: [T_0, \infty)$ ,  $T_0 \geq 0$ , and  $h_i \in C$  on  $L \times E^n$ ,  $h_i(t, 0) \equiv 0$  on  $L$ ,  $i = 1, 2$ . Assume further that  $h_i$  is Lipschitzian on  $L \times E^n$  for a function  $\gamma_i(t) \geq 0$  with  $\gamma_i(t) \rightarrow 0$  as  $t \rightarrow \infty$  and  $\int_{T_0}^{\infty} \gamma_i(t)dt < \infty$ . A function  $\varphi(t, \eta) \in C$  on  $L \times I$ ,  $I: (0, \Delta)$ , non-decreasing in  $\eta$  on  $L$ , is a [strong] comparison function, of rank  $m$ , for (\*) if, whenever there are  $m$  eigenvalues of  $A$  with real parts  $\leq \rho_p$ , for every  $\eta \in I$  there are constants  $\alpha > \rho_p$ ,  $\beta < \rho_{p+1}$  such that  $\alpha \leq [\varphi(t', \eta) - \varphi(t'', \eta)] / (t' - t'') \leq \beta$  for  $t' \neq t''$ ,  $t', t'' \in L$  [and if, for every  $\eta \in I$ ,  $\varphi(t, \eta) - \varphi(t, \eta/2) \rightarrow +\infty$  with  $t$ ].

The author proves that if  $\varphi$  is a [strong] comparison function, of rank  $m$ , for (\*) and if  $\int_{T_0}^{\infty} \|\dot{f}\| \exp[-\varphi(t, \eta)]dt < \infty$  on  $I$ , then there is an  $m$ -parameter family,  $\mathcal{F}$ , of solutions such that, for any  $\eta \in I$ ,  $\|x(t)\| \exp[-\varphi(t, \eta)] = O(1)$  [ $O(1)$ ] as  $t \rightarrow \infty$  holds only for every  $x(t) \in \mathcal{F}$ . H. A. Antosiewicz.

**Markus, Lawrence; and Moore, Richard A.** Oscillation and disconjugacy for linear differential equations with almost periodic coefficients. Acta Math. 96 (1956), 99-123.

The differential equation  $(r(x)y')' + K(x)y = 0$ , in which  $r(x) > 0$  and  $K(x)$  are real continuous functions on  $-\infty < x < \infty$ , is said to be non-oscillatory if every non-trivial solution  $y$  is non-vanishing for  $|x|$  sufficiently large (otherwise, oscillatory). A non-oscillatory equation is disconjugate if every non-trivial solution has at most one zero on  $-\infty < x < \infty$ . Let  $p(x)$  be a real continuous function on  $-\infty < x < \infty$  and let  $a, b$  denote real parameters. The disconjugacy domain  $D$  of the equation

$$(*) \quad y'' + (-a + bp(x))y = 0$$

is defined as the set of points  $(a, b)$  of the real plane for which (\*) is disconjugate.

Various results concerning the domain  $D$  of (\*) in case  $p(x)$  is a real almost periodic function are obtained. It is shown, among other things, that (i)  $D$  is closed and convex; (ii) if  $p(x) \not\equiv 0$  has mean zero, the set  $D$ , except for the origin, is contained in the half-plane  $a > 0$ ; (iii) if  $\int_0^x p(s)ds$  is almost periodic then  $D$  has a boundary  $a = a(b)$  which is tangent to the  $b$  axis at the origin.

The authors generalize certain parts of the theory of the Hill equation (equation (\*) in which  $p(x)$  is periodic). In particular, it is shown that if  $(a, b)$  is in the interior of the set  $D$  belonging to (\*) and if  $p(x)$  is real and almost periodic, then (\*) has a pair of independent solutions each having an almost periodic logarithmic derivative.

Further results relating to the interior, as well as to the boundary, of the set  $D$  are also obtained. In addition there is investigated the effect on the set  $D$  of perturbations in the function  $p(x)$ . C. R. Putnam.

**Švec, Marko.** Eine Eigenwertaufgabe der Differentialgleichung  $y^{(n)} + Q(x, \lambda)y = 0$ . Czechoslovak Math. J. 6(81) (1956), 46-71. (Russian summary)

The author considers the oscillations of the solutions of

$$(a) \quad y^{(n)} + Q(x)y = 0,$$

where  $Q(x)$  is a non-negative real continuous function on  $(-\infty, \infty)$  which vanishes on no subinterval. Typical of a number of results is the following theorem. Let  $y(x)$  be a real solution of (a) satisfying  $y^{(j)}(x_1) > 0$  for  $j = 0, 1, 2, \dots, k-1$ ;  $y^{(k)}(x_1) \neq 0$ ; and  $y^{(j)}(x_1) = 0$  for  $j = k+1, k+2, \dots, n-1$  for some point  $x_1$  and integer  $k$  on  $0 \leq k \leq n-1$ . Then at each point  $\xi > x_1$  at most one of the functions  $y^{(j)}(x)$ ,  $j = 0, 1, 2, \dots, n-1$ , is zero. If  $y^{(m)}(\xi) = 0$ ,  $0 \leq m \leq n-1$ , then  $y^{(j)}(\xi) \neq 0$  for  $j = 0, 1, 2, \dots, m-1, m+1, \dots$  and also the  $y^{(j)}(\xi)$  all are of the same sign for  $j > m$  (also for

$j < m$ ). There are refinements for the cases of  $n$  even or odd.

Now let  $M_{ik}$  be the set of real solutions of (a) satisfying  $y^{(j)}(x_1) = 0$ ,  $j = 0, 1, 2, \dots, i-1, i+1, \dots, k-1, k+1, \dots, n-1$ , for integers  $0 \leq i < k \leq n-1$ . The set of functions  $M_{ik}$  can then be described as the set of all solutions of a certain second order linear homogeneous differential equations. Thus the standard oscillation theory applies to the functions of  $M_{ik}$ .

The author then applies his analysis to the eigenvalue problem for the equation

$$(b) \quad y^{(n)} + Q(x, \lambda)y = 0$$

for real solutions satisfying the boundary conditions

$$y^{(j)}(x_1) = 0,$$

$$(j = 0, 1, \dots, i-1, i+1, \dots, k-1, k+1, \dots, n-1;$$

$$0 \leq i < k \leq n-1)$$

$$y(x_2) = 0, y(x_3) = 0,$$

where either (a)  $x_1 \leq x_2 < x_3$  or (b)  $x_3 < x_2 \leq x_1$ , and  $x_1, x_2, x_3$  are points on the finite interval  $(a, b)$ . Theorem. Let  $Q(x, \lambda)$  be continuous in  $(x, \lambda)$  for  $x \in [a, b]$  and  $\lambda \in [\Delta_0, \Delta_1]$ . Let  $Q(x, \Delta_0) = 0$ ,  $Q(x, \lambda) > 0$  for  $\Delta_0 < \lambda < \Delta_1$  and assume  $\lim_{\lambda \rightarrow \Delta_1} Q(x, \lambda) = \infty$  uniformly for  $x \in [a, b]$ . Let  $n \geq 4$  be even. Then the eigenvalue problem (a), also (b), has infinitely many real eigenvalues, all simple, which form a sequence  $\Delta_0 < \lambda_1^{(k)} < \lambda_2^{(k)} < \dots < \Delta_1$ . The eigenfunction  $y(x, \lambda_r^{(k)})$  has in the interval  $(x_2, x_3)$  in case (a), or in  $(x_3, x_2)$  in case (b), exactly  $(r-1)$  zeros each of which is simple. There is a corresponding result when  $n \geq 3$  is odd.

L. Markus (Princeton, N.J.).

\*Erugin, N. P. Metod Lappo-Danilevskogo v teorii lineinykh differentsial'nykh uravnenii. [The method of Lappo-Danilevskii in the theory of linear differential equations.] Izdat. Leningr. Univ., 1956. 108 pp. 3.50 rubles.

The Soviet mathematician Lappo-Danilevskii, who died in 1931, aged 35, and to whom this monograph is dedicated, left a noteworthy volume [Trudy Fiz.-Mat. Inst. Steklov. 6 (1934)] primarily devoted to analytic functions of matrices. In the last two chapters the following problem is dealt with. Let

$$(1) \quad \frac{dX}{dz} = XP(z)$$

be a differential equation in the matrix  $X$  of order  $n$ , with  $P$  a matrix of the same order regular analytic around  $z=a$  and with a pole or essential singularity at  $a$ . As  $z$  turns around  $a$  once a solution  $X(z)$  becomes  $\tilde{X}(z) = V(a)X$  where  $V(a)$  is a constant matrix. If  $W(a) = (2\pi i)^{-1} \log V$ , then  $Y(z) = (z-a)^W N(z)$ , where  $N(z)$  is single-valued in the neighborhood of  $a$ . Consider in particular the system (finite sum)

$$(2) \quad \frac{dX}{dz} = X \sum \frac{U_j}{z-a_j}.$$

This system was studied by Poincaré [Acta Mat. 4 (1884), 201-312] who dealt with the general form of the matrices  $V_j$  corresponding to each  $a_j$  in terms of the  $U_j$ . The same problem was studied by Mittag-Leffler [ibid. 15 (1891), 1-32] when  $\{a_j\}$  is countable but without finite condensation point. Lappo-Danilevskii gave the explicit expression of the functions  $V_j(U_1, \dots, U_m)$  for (2), and likewise of the function  $V$  when  $P(z)$  has a multiple pole at  $a$ .

In the present monograph the author considers the

same problems and contributes a number of interesting complements and applications. He deals then with the real analogue of (1)

$$(3) \quad \dot{X} = XP(t),$$

where  $P(t)$  is continuous and has the period  $2\pi$ . There are here the analogues of  $V, W, N$  save that they correspond to  $t$  varying by  $2\pi$  and  $N$  is periodic. Consider particularly the case

$$P(t) = a_0 + \sum (a_k \sin kt + b_k \cos kt),$$

where the sum is finite. Make the change of variable  $z = \exp it$ . As a consequence (3) becomes

$$(4) \quad \frac{dX}{dz} = X \sum_{k=-m}^{+m} P_k z^{k-1},$$

where the  $P_{\pm k}$  are readily expressed in terms of  $a_k, b_k$ . Here the sum has the origin as only singularity: a pole, and one may apply the results of Lappo-Danilevskii, hence obtain  $V, W, N$  for (3) itself. The procedure is applied to Mathieu's equation. S. Lefschetz.

\*Koosis, Paul. One-dimensional repeating curves in the non-degenerate case. Contributions to the theory of nonlinear oscillations, vol. 3, pp. 277-285. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. \$4.00.

The author proves that the system

$$\dot{x}_1 = 1 + x_3 P_1(x_1, x_2, x_3) + k Q_1(x_1, x_2, x_3) \quad (i=1, 2),$$

$$\dot{x}_3 = x_3 [A(x_1, x_2) + x_3 P_3(x_1, x_2, x_3)] + k Q_3(x_1, x_2, x_3),$$

where the  $x_j$ 's are scalars,  $A, P_j, Q_j$  are of class  $C^1$  and of period  $\omega_1$  in  $x_1$ , and  $\int_0^{\omega_1} A(s+t, t) dt \neq 0$  for all  $s$ , possesses near  $k=0$  solutions that form a surface which is invariant under the translations  $x_1 \rightarrow x_1 + n\omega_1, x_3 \rightarrow x_3, n$  an integer, and which, as  $k \rightarrow 0$ , tends to the plane  $x_1 = s+t, x_2 = t, x_3 = 0$ . H. A. Antosiewicz (Washington, D.C.).

\*Diliberto, Stephen P. Bounds for periods of periodic solutions. Contributions to the theory of nonlinear oscillations, vol. 3, pp. 269-275. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. \$4.00.

The author obtains bounds for the period of a periodic solution of the two dimensional system of real first order equation  $dx/dt = X(x)$ ,  $X \in C^1$  for all  $x$ . Using the fact that the periods are bounded, he obtains a uniform convergence theorem for periodic solutions of  $dx/dt = X(x, \lambda)$  to a solution of  $dx/dt = X(x, 0)$ . This convergence theorem is then used to prove a theorem on rotated vector fields.

J. K. Hale (Albuquerque, N.M.).

Marčenko, V. A. On finite perturbations of one-dimensional differential operators of second order. Har'kov. Gos. Univ. Uč. Zap. 40=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 23 (1952), 73-77 (1954). (Russian)

Let  $q$  be a real-valued function defined on  $0 \leq x < a$  ( $a \leq \infty$ ), and let  $L$  denote the formal differential operator  $Lu = u'' - qu$  on  $0 \leq x < a$ . If  $\hat{q}$  vanishes in a neighborhood of  $a$ , then  $\hat{L}$ , where  $\hat{L}u = Lu - \hat{q}u$ , is called a finite perturbation of  $L$ . In this case  $L$  and  $\hat{L}$  will both be in the limit circle case at  $a$ , or will both be in the limit point case at  $a$ . Let  $\Lambda(L, h) [\Lambda(\hat{L}, h)]$  denote the set of eigenvalues for the problem  $Lu + \lambda u = 0, u'(0) = hu(0) [\hat{L}u + \lambda u = 0,$

$u'(0) = hu(0)$ ], together with the same boundary condition at  $a$ , if  $L$  and  $\hat{L}$  are in the limit circle case at  $a$ . Let  $n(x)$  be the number of points of the set  $\Lambda(L, h) \cap \Lambda(\hat{L}, \hat{h})$  in the interval  $-x < \lambda < x$ . The author proves that, if  $\hat{q}$  is not identically zero, and  $\hat{q}(x) = 0$  for  $x > b$ ,  $b > a$ , then

$$\limsup_{x \rightarrow \infty} x^{-1} n(x) \leq 2b/\pi.$$

E. A. Coddington (Los Angeles, Calif.).

**Razumihin, B. S.** On stability of systems with retardation. Prikl. Mat. Meh. 20 (1956), 500-512. (Russian)

The author proves various sufficient criteria for the simple and asymptotic stability of the trivial solution of the system  $\dot{x}_1 = X_1(t; x_1(t), x_2(t-\tau))$ ,  $1 \leq i, j \leq n$ ,  $1 \leq k \leq m$ , where  $X_i$  is continuous and bounded on  $t \geq 0$ ,  $|x_i| < H$ ,  $X_i(t; 0, 0) = 0$  on  $t \geq 0$ , the  $x_{jk}$  are certain functions, and the parameter  $\tau$  is restricted to  $0 \leq \tau \leq h_{jk}(t)$  with  $h_{jk}(t)$  positive and bounded on  $t \geq 0$ . These criteria are all based upon considerations of positive definite functions in the manner of Krasovskii's generalization [Prikl. Mat. Meh. 20 (1956), 315-327; MR 18, 128] of Lyapunov's second method. A number of examples is discussed.

H. A. Antosiewicz (Washington, D.C.).

**Bauer, W.** Darstellung von Einflusszahlen in Matrizen-schreibweise. Z. Angew. Math. Mech. 36 (1956), 272.

Nach der bekannten Methode der Matrizenrechnung zur Behandlung von Schwingungs- und Stabilitätsproblemen wird für lineare Differentialgleichungen die Lösung der Anfangswertaufgabe in der Form: Zustandsvektor = Übertragungsmatrix · Anfangsvektor angegeben. In Sonderfällen haben die Übertragungsmatrizen die Eigenschaften der „Rayleigh-Matrix“.

R. Gran Olsson (Trondheim).

**Ráb, Miloš.** Oscillatorische Eigenschaften der Lösungen linearer Differentialgleichungen 3. Ordnung. Acta Acad. Sci. Českoslovenicae Basis Brunensis 27 (1955), 349-360. (Czech. Russian and German summaries)

Im ersten Teil werden die oscillatorischen Eigenschaften der Lösungen der selbstadjungierten Differentialgleichung (1)  $y''' + 2A(x)y' + A'(x)y = 0$  im Zusammenhang mit den Lösungen von (2)  $y'' + \frac{1}{2}A(x)y = 0$  untersucht. Besonders wird eine notwendige und hinreichende Bedingung für die Oszillation eines willkürlichen Integrals von (1), eine Abschätzung für die Entfernung der Nullstellen und ein Vergleichssatz bewiesen. Im zweiten Teil befasst sich der Autor mit der linearen Differentialgleichung dritter Ordnung in der kanonischen Form  $y''' + 2A(x)y' + [A'(x) + \omega(x)]y = 0$ , wo  $\omega(x) \geq 0$  und beweist eine notwendige und hinreichende Bedingung für die Oszillation eines willkürlichen Integrals und einen Trennungssatz.

M. Zlámal (Brno).

★ **Diliberto, S. P.; and Marcus, M. D.** A note on the existence of periodic solutions of differential equations. Contributions to the theory of nonlinear oscillations, vol. 3, pp. 237-241. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. \$4.00.

Consider  $\dot{x} = A(t)x + p(x, t) + q(x, t, k)$ , where  $x \in E^n$ ,  $k \in E^1$ , and  $A, p, q \in C$  for all values of their arguments, periodic in  $t$  of period  $T$ , and  $p, q$  are locally Lipschitzian in  $x$ . Assume  $\|p(x, t)\| \leq K\|x\|^2$  for  $\|x\| \leq R$ ,  $t \in E^1$ , and

$q(x, t, 0) = 0$ . The authors prove — Diliberto by use of Borsuk's theorem, Marcus by direct methods and under the further hypothesis that  $p \in C^3$  on  $E^n \times E^1$  — that if the characteristic exponents of  $\dot{x} = A(t)x$  have non-zero real parts there exists for all  $k$  with  $|k|$  sufficiently small a solution of period  $T$ . This sharpens earlier results of Farnell, Langenhop and Levinson [J. Math. Phys. 29 (1951), 300-302; MR 12, 706] and the reviewer [Ann. of Math. (2) 57 (1953), 314-317; 58 (1953), 592; MR 14, 751; MR 15, 224].

H. A. Antosiewicz.

**Krasovskii, N. N.** On asymptotic stability of systems with after-effect. Prikl. Mat. Meh. 20 (1956), 513-518. (Russian)

In continuation of his earlier work [Prikl. Mat. Meh. 20 (1956), 315-327; MR 18, 128] the author considers equations  $\dot{x} = X(x(t-\theta), t)$  with  $X(0, t) = 0$ , defined on  $\|x(t-\theta)\| < H$ ,  $t \geq 0$ , and proves two criteria in order that (i), given any  $\epsilon > 0$ , there exist a  $\delta > 0$  such that whenever  $\|x_0(t_0 - \theta_0)\| < \delta$  then  $\sup\{\|x_1(x_0(t_0 - \theta_0), t - \theta)\| : 0 \leq \theta \leq h\} < \epsilon$  for  $t \geq t_0$ , and (ii)  $x(x_0(t_0 - \theta_0), t) \rightarrow 0$  as  $t \rightarrow +\infty$  for every  $\|x_0(t_0 - \theta_0)\| < G \leq H$ . These criteria rest upon properties, too involved to be stated here, of positive definite functions which play the role of Lyapunov functions.

H. A. Antosiewicz (Washington, D.C.).

**Bedel'baev, A. K.** On a construction of the function of Lyapunov as a quadratic form. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. 1956, no. 4(8), 24-37. (Russian)

Moiseev [Notes of a seminar on the theory of stability of motion, 1948] gave an explicit construction of a Lyapunov quadratic form for a system of order  $n$  with constant coefficients. An alternate construction is presented in this Note, which is claimed to be more manageable for large  $n$ .

S. Lefschetz (Princeton, N.J.).

**Santoro, Paolo.** Un criterio di limitatezza in futuro delle soluzioni di una equazione differenziale non lineare. Boll. Un. Mat. Ital. (3) 11 (1956), 432.

Theorem: Consider the equation

$$(1) \quad x'' + \omega(x, x')x' + h(x) = p(t),$$

where  $\omega(x, x')$ ,  $h(x)$  are continuous Lipschitzian functions in some finite region and  $p(t)$  is continuous in  $(t_0, +\infty)$ . If  $\omega(x, x') = \varphi(x, x') + \psi(x, x')$ , (i)  $\varphi(x, x') > 0$  (ii) there is a function  $f(x)$  such that

$$\varphi(x, x') \geq f(x)a^{-1}(x)x', \quad a(x) = \exp\left(\int_0^x f(u)du\right),$$

$0 < a(x) < A$  ( $A$  constant), (iii)  $H(x) = \int_0^x a^2(u)h(u)du > 0$ ,  $x \neq 0$  and  $\lim_{|x| \rightarrow +\infty} H(x) = +\infty$ , and (iv)  $\int_0^x |\dot{p}(t)|dt < \infty$ , then every solution of (1), together with all its derivatives, is bounded for all  $t$ . This extends a result of Antosiewicz [J. London Math. Soc. 30 (1955), 64-67; MR 16, 477].

J. K. Hale (Albuquerque, N.M.).

**Goršin, S. I.** On stability with a countable number of perturbations in a certain critical case. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. 1956, no. 4(8), 38-42. (Russian)

A certain generalized type of stability has been introduced and dealt with by Goršin [See MR 14, 48, 752]. It may be described as follows: Take a system in an  $n$ -vector  $x$ :

$$(1) \quad \dot{x} = \omega(t; x) + f(t; x),$$

where in a certain set  $E: t \geq 0, \|x\| \leq H$  (norm =  $\sup |x_i|$ ), we have  $\omega(t; 0) = 0$ ,  $\omega$  continuous and  $f$  is to be chosen arbitrarily such that  $\|f\| < \rho \leq H$ ; it is also assumed that the usual conditions, unicity included, for the solutions of (1) hold in  $E$ . Stability of the solutions of (1) of various kinds at the origin is now defined as follows: For every  $\varepsilon$  and  $t_0$  there exist  $\eta(\varepsilon, t_0)$  and  $\rho(\varepsilon, t_0)$  such that if  $\|x_0\| < \eta$  and  $\|f(t)\| < \rho$  then the solution  $x(t)$  such that  $x(t_0) = x_0$  satisfies  $\|x(t)\| < \varepsilon$  for all  $t \geq t_0$ . Asymptotic and uniform stability likewise instability are defined in the natural way. Goršin has extended many of the known results of the stability theory to this type even when the system is countable.

On the other hand Persidskiĭ [see MR 14, 753] has considered a system in two vectors  $x, y$ :

$$(2) \quad \begin{cases} (a) & \dot{x} = X(t, x, y) \\ (b) & \dot{y} = Y(t, x, y) \end{cases}$$

and defined quasi-stability (of various types) of (2a) under the condition that  $y$  be replaced by  $g(t)$ , small enough, and likewise for (2b) with  $x, y$  interchanged. He also proved that quasi-stability of both (2a) and (2b) implies stability for (2). In the present paper the author combines the two points of view, for systems such as (2) with  $X, Y$  replaced  $X+f, Y+g$ , with  $(f, g)$  dealt with more or less like  $f$  previously, and obtains a certain number of extensions. His systems are countable. *S. Lefschetz.*

**Persidskiĭ, S. K.** On the second method of Lyapunov. *Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh.* 1956, no. 4(8), 43-47. (Russian)

Consider an  $n$ -vector system

$$(1) \quad \dot{x} = f(t; x), \quad f(t; 0) = 0 \text{ for } t \geq 0,$$

where  $f$  and its partials as to the  $x_i$  are continuous for  $t \geq 0$  wherever considered. In analogy to Lyapunov we have: Theorem 1. Let there exist a function  $V$  defined in a closed spherical region  $S$  centered around the origin and of radius  $\rho$ , where (a)  $V > 0$  outside the origin; (b)  $\dot{V} \geq 0$ ; (c)  $V \rightarrow 0$  as  $t \rightarrow +\infty$  uniformly in  $x$ . Then the origin is completely unstable for (1). Theorem 2. (Converse of 1). If the origin is completely unstable a  $V$  exists such as described. *S. Lefschetz (Princeton, N.J.).*

**Kurzeil, Jaroslav.** On oscillations of autonomous non-linear systems of differential equations. *Czechoslovak Math. J.* 5(80) (1955), 517-531. (Russian. English summary)

The author points out that if the linear autonomous  $[A$  is a constant matrix] system  $\dot{x} = Ax$  has a periodic solution  $x_0(t)$ , then sufficient conditions are known in order that the "nearly linear" autonomous system  $\dot{x} = Ax + \mu f(x, \mu)$  have a periodic solution that approaches  $x_0(t)$  uniformly as  $\mu$  approaches 0. The oldest theorem of this type assumes that  $f(x, \mu)$  is analytic in  $(x, \mu)$  [Poincaré's method of small parameters]. The author replaces analyticity by a Lipschitz condition and establishes — by a method of successive approximations — a similar result. The paper is well written and the proof is interesting. [For a somewhat more general result of this type see Coddington and Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955, Ch. 14; MR 16, 1022.] *J. P. LaSalle (Notre Dame, Ind.).*

**Conti, Roberto.** Limitazioni "in ampiezza" delle soluzioni di un sistema di equazioni differenziali e applicazioni. *Boll. Un. Mat. Ital.* (3) 11 (1956), 344-349.

Theorem: Let  $u_0(t)$  be the maximum solution of the

equation  $u' = \omega(t, u)$ ,  $u_0(t_0) = u_0$ ,  $\omega(t, u)$  continuous in  $a < t < b$ ,  $0 < u < +\infty$ , and let  $T^+$  be the supremum of the values of  $t$  for which  $u_0(t)$  is defined. If  $\rho = (\sum_{i=1}^n |x_i|^2)^{1/2}$ ,  $x = (x_1, \dots, x_n)$ ,  $f = (f_1, \dots, f_n)$  and

$$\bar{x}^T f(t, x) + f^T(t, x)x \leq 2\rho\omega(t, \rho)$$

( $a < t < b$ ), then the solution  $x(t)$  of  $x' = f(t, x)$ ,  $x(t_0) = x_0$ ,  $(\sum_{i=1}^n |x_i(t)|^2)^{1/2} = u_0$ , satisfies  $\rho(t) \leq u_0(t)$ ,  $t_0 \leq t < T^+$ ,  $\rho(t) = (\sum_{i=1}^n |x_i(t)|^2)^{1/2}$ . A similar theorem is stated concerning the minimum solution  $v_0(t)$  of  $v' = \alpha(t, u)$ . Using these results, the author proves the well-known result of Ważewski [Studia Math. 10 (1948), 48-59; MR 10, 40] that if  $f(t, x) = A(t)x$ ,  $\lambda(t)[\Lambda(t)]$  is the least [largest] characteristic value of  $\frac{1}{2}A + \frac{1}{2}A^T$ , then

$$\rho(t) \exp\left(\int_{t_0}^t \lambda(\tau) d\tau\right) \leq \rho(t) \leq \rho(t_0) \exp\left(\int_{t_0}^t \Lambda(\tau) d\tau\right).$$

A corollary of the above theorem is: If  $x^T f(t, x) + f^T(t, x)x \leq 2\rho\mu(t)h(u)$  and  $\int_{t_0}^{+\infty} [h(u)]^{-1} du = +\infty$ , then every solution of  $x' = f(t, x)$  can be extended in  $t$ . This generalizes a result of Wintner [Amer. J. Math. 68 (1946), 173-178; MR 7, 297]. The author also obtains boundedness theorems using the previous theorems. *J. K. Hale.*

**Rutovitz, D.** On the  $L_p$ -convergence of eigenfunction expansions. *Quart. J. Math. Oxford Ser. (2)* 7 (1956), 24-38.

It is first proved that the Sturm-Liouville expansion of a function  $f(x)$  of class  $L_p$  over a finite interval, arising from the equation  $\{D^2 + \lambda - q(x)\}y = 0$  with boundary conditions at the endpoints, converges in mean with index  $p$  to  $f(x)$ . A similar result is then obtained in the case where the interval is  $(0, \infty)$  if  $xq(x)$  is integrable and of bounded variation over  $(0, \infty)$ . *E. C. Titchmarsh (Oxford).*

**Crum, M. M.** On certain Sturm-Liouville functions. *J. London Math. Soc.* 31 (1956), 426-432.

Let  $\phi_\alpha(s, x)$  be the solution of  $\phi'' + (s^2 - q_\alpha(x))\phi = 0$  such that  $\phi(0) = 1$ ,  $\phi'(0) = 0$ , where  $q_\alpha(x)$  is continuous in  $[0, \pi]$ ; likewise define  $\phi_\beta(s, x)$  corresponding to  $q_\beta(x)$ . The author uses a Fourier transform method to prove the existence of a kernel  $K_\alpha(x, z)$  such that

$$\phi_\alpha(s, x) = \cos sx + \int_0^x K_\alpha(x, z) \cos sz dz.$$

Hence he establishes an expansion of an arbitrary  $f(x) \in L(0, \pi)$  in a series of the  $\phi_\alpha(s_n, x)$ , where the  $s_n$  are eigen-values given by  $\phi_\beta'(s_n, \pi) = 0$ , proving equi-convergence with the Fourier cosine-series of  $f(x)$ . There is a formal connection with Schlömilch series [see, e.g., Watson, *Theory of Bessel functions*, 2nd ed., Cambridge, 1944, Ch. 19; MR 6, 64]. [Reviewer's remarks: A more extensive theory of integral transformations between  $\phi(s, x)$  and  $\cos sx$  has been given by A. Povzner [Mat. Sb. N.S. 23(65) (1948), 3-52; Ch. II; MR 10, 299; 11, 360 and B. M. Levitan [Uspehi Mat. Nauk (N.S.) 4 (1949), no. 1(29), 3-112, Ch. V; MR 11, 116; 13, 463], not discussed by the present author. Povzner and Levitan use an approach based on partial differential equations, which in Crum's work appears to be incidental.] *F. V. Atkinson.*

**Čudov, L. A.** A new variant of an inverse Sturm-Liouville problem on a finite interval. *Dokl. Akad. Nauk SSSR (N.S.)* 109 (1956), 40-43. (Russian)

Let the spectra  $\{\lambda_n\}$ ,  $\{\bar{\lambda}_n\}$ , of  $-y'' + q(x)y = \lambda y$  with the two sets of boundary conditions (1)  $y'(0, \lambda) = h_0 y(0, \lambda)$ ,  $y'(1, \lambda) = h_1 y(1, \lambda)$ , and (2)  $y'(0, \lambda) = \bar{h}_0 y(0, \lambda)$ ,  $y'(1, \lambda) =$

$\tilde{h}_1(1, \lambda)$  coincide with the spectra of  $-z'' + q_1(x)z = \lambda z$  under the same conditions. The question is the validity of the uniqueness conclusion  $q=q_1$ . Several proofs have been given of this for the case  $h_0=\tilde{h}_0$ ,  $h_1 \neq \tilde{h}_1$ , in particular by the author [Mat. Sb. N.S. 25(67) (1949), 451-456, which see for other references; MR 11, 248]. Relaxing the condition  $h_0=\tilde{h}_0$ , the result of this paper appears to be that  $q=q_1$  holds if  $|\tilde{h}_0 - h_0|/|\tilde{h}_1 - h_1|$  is either sufficiently small or sufficiently large. The proof uses a partial differential equation method to establish integral relations concerning the  $y(x, \lambda)$ ,  $z(x, \lambda)$  (cf. references at end of preceding review). It depends on the completeness of the functions  $y(x, \lambda_n)z(x, \lambda_n)$  and  $y(x, \tilde{\lambda}_n)z(x, \tilde{\lambda}_n)$ ; though it is pointed out that there are cases in which they are not complete.

F. V. Atkinson (Canberra).

**Kalafati, P. On a new orthogonal system of functions.**

Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 631-633. (Russian)

Consider the equation  $y'' - V(x)y + \lambda^2 y = 0$  ( $0 \leq x \leq 1$ ) with boundary conditions  $y'(0) - hy(0) = y(1) = 0$ , where  $V(x)$  is measurable, integrable and real-valued and  $h$  real. Let the eigenvalues  $\lambda_k^2$  ( $k=1, 2, \dots$ ) be assumed positive, take  $\lambda_k > 0$  and define  $\lambda_{-k} = -\lambda_k$ . Let  $\phi(x; \lambda^2)$  be the solution such that  $\phi(0; \lambda^2) = 1$ ,  $\phi'(0; \lambda^2) = h$ . A complete orthogonal system over  $(0, 1)$  is then given by

$$\chi_k(x) = \left[ \phi(x; \lambda_k^2) + i\lambda_k^{-1} \phi(x; 0) \frac{d}{dx} \frac{\phi(x; \lambda_k^2)}{\phi(x; 0)} \right] \exp(-i\lambda_k x),$$

( $k = \pm 1, \pm 2, \dots$ ). The proof is based on the study of an integral equation, with skew-symmetric Hermitean kernel.

F. V. Atkinson (Canberra).

**Levitan, B. M.; and Sargsyan, I. S. Theorem on convergence of twice differentiated expansion according to eigenfunctions of a Sturm-Liouville operator.** Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki 9 (1956), no. 3, 3-15. (Russian. Armenian summary)

The differential system

$$y'' + (\lambda - q)y = 0, \quad y(0) \cos \alpha + y'(0) \sin \alpha = 0$$

is considered on  $0 \leq x < \infty$ . The following result is stated: Let  $q$  have a summable second derivative on each finite interval, suppose  $f, f', f'' - qf$  are in  $L_2(0, \infty)$  and

$$\lim_{x \rightarrow \infty} [f(x)E_\lambda'(x) - f'(x)E_\lambda(x)] = 0,$$

$$E_\lambda(x) = \int_0^\lambda \varphi(x, \lambda) d\rho(\lambda) \quad (\lambda \neq 0).$$

If  $g(\lambda) = \int_0^\infty f(x)E_\lambda(x)dx$ , then  $f''(x) = \int_{-\infty}^\infty \varphi''(x, \lambda) dg(\lambda)$ . Here, presumably,  $\varphi$  is a solution of the differential system, and  $\rho$  is a spectral function. In a letter to the editor, which appeared later in the same journal, the authors state that to this theorem should be added a further assumption that  $f''$  should satisfy a local condition at  $x$  sufficient to guarantee its expansion in an ordinary Fourier series. Then the integral given for  $f''$  is equi-convergent with the Fourier cosine integral expansion of  $f''$ . Another error pointed out in this letter is that in formula (4) on page 5 only the first terms for  $w_{\pm}^{(k)}(x, t, s)$  are given, but the authors claim this omission does not affect the results. The theorem stated above is proved only in the case of discrete spectra; it is stated that the general case follows by analogy.

E. A. Coddington (Los Angeles, Calif.).

**Kac, I. S. On the existence of spectral functions of certain second-order singular differential systems.** Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 15-18. (Russian)

The author considers generalized forms of singular eigenvalue problems for second order differential operators, and gives sufficient conditions under which these problems have a simple spectrum. Let  $M$  be a non-decreasing function defined in  $a \leq x < b$  or  $a < x < b$  ( $-\infty \leq a < b \leq \infty$ ), and suppose  $Q$  is a real-valued function, defined on one of these two intervals, which is of bounded variation on each closed segment  $[\alpha, \beta] \subset (a, b)$ . The point  $a$  is called regular if  $a > -\infty$ ,  $M$  and  $Q$  are defined at  $a$ , and  $Q$  is of bounded variation on each segment  $[\alpha, \beta]$ , where  $a < \beta < b$ . Otherwise  $a$  is called singular. The notation  $a < t$  is used to mean  $a \leq t$  or  $a < t$  according as  $a$  is a regular point or not. If  $a$  is regular, a function  $f$  on  $[a, b)$  is an ordinary function defined there together with a number  $f^-(a)$  (left derivative at  $a$ ). By  $D_M Q$  is denoted the set of all functions  $f$  such that: (a)  $f$  is absolutely continuous on each closed segment  $[\alpha, \beta]$ ,  $a < \alpha < \beta < b$ ; (b)  $f$  has a finite left derivative  $f^-(x)$  at each  $x$ ,  $a < x < b$ ; (c) the function  $f^{(1)}$  given by

$$f^{(1)}(x) = f^-(x) - \int_a^x f(s) dQ(s) \quad (a < x < b)$$

is  $M$ -absolutely continuous on each segment  $[\alpha, \beta]$ ,  $a < \alpha < \beta < b$ . For  $f \in D_M Q$  the function  $l[f] = -df^{(1)}/dM$  exists  $M$ -almost everywhere on  $a < x < b$ . A function  $u$  is said to be a solution of  $l[y] = q$  if  $u \in D_M Q$  and  $l[u] = q$  is valid  $M$ -almost everywhere on  $a < x < b$ . Let  $\mathfrak{F}$  be the set of all  $M$ -measurable functions which are square summable with respect to  $M$  on  $a < x < b$ , and let  $\mathfrak{D}$  be the subset of those  $f \in \mathfrak{F} \cap D_M Q$  for which  $l[f] \in \mathfrak{F}$ . Let  $L$  be the operator in  $\mathfrak{F}$  with domain  $\mathfrak{D}$ , and such that  $Lf = l[f]$ ,  $f \in \mathfrak{D}$ . The operator  $L_0 = L^*$  is symmetric. Corresponding to any self-adjoint extension of  $L_0$  there is a two-by-two spectral matrix function  $\tau$  (on  $-\infty < \lambda < \infty$ ), and there exists an isometry between  $\mathfrak{F}$  and  $\mathfrak{L}^2(\tau)$ . The author is interested in the problem when this matrix can be considered to be one-by-one, i.e., a scalar spectral function. In this case the spectrum is simple. According to M. G. Kreĭn, if the point  $a$  is regular then the system  $l[y] = \lambda y$  ( $a \leq x < b$ ),  $y(a) = m$ ,  $y^-(a) = n$  ( $m, n$  real numbers) has a unique solution, and there exists at least one orthogonal spectral function, which is generated by some self-adjoint extension of  $L_0$ . The spectral function  $\tau$  is orthogonal if the usual mapping from  $\mathfrak{F}$  to  $\mathfrak{L}^2(\tau)$  is unitary. The author quotes two results (without proofs) which guarantee the existence of self-adjoint extensions of  $L_0$  with simple spectrum in the case where there are two singular ends. Theorem I. If  $a=0$ ,  $Q(x) = -(\nu^2 - \frac{1}{4})x^{-1} + Q_1(x)$ ,  $\nu > 0$ , and  $\int_0^\infty x |dQ_1(x)| < \infty$ ,  $\int_0^\infty x dM(x) < \infty$  ( $0 < h < b \leq \infty$ ), then the system

$$l[y] = \lambda y \quad (0 < x < b), \quad \lim_{x \rightarrow 0} x^{-\nu-1} y(x) = 1,$$

has a unique solution which is entire in  $\lambda$  for fixed  $x$ , and there exists at least one orthogonal spectral function which is generated by a self-adjoint extension of  $L_0$ . If, in addition,  $\int_0^\infty x^{1-2\nu} dM(x) = \infty$  ( $0 < h < b$ ), then any self-adjoint extension of  $L_0$  generates an orthogonal spectral function. Theorem II. If  $a = -\infty$ ,  $\int_{-\infty}^b |s| dM(s) < \infty$ ,  $\int_{-\infty}^b |s| |dQ(s)| < \infty$  ( $-\infty < h < b \leq \infty$ ), then the system

$$l[y] = \lambda y \quad (-\infty < x < b), \quad \lim_{x \rightarrow -\infty} y(x) = 1,$$

has a unique solution which is entire in  $\lambda$  for fixed  $x$ , there exists at least one orthogonal spectral function, and if in

addition  $\int_{-\infty}^{\infty} x^2 dM(x) = \infty$ , then every self-adjoint extension of  $L_0$  generates an orthogonal spectral function.

E. A. Coddington (Los Angeles, Calif.).

**Kac, I. S. On the behaviour of spectral functions of second-order differential systems.** Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 183-186. (Russian)

Let  $M$  be a non-decreasing function on  $0 \leq x < B \leq \infty$ , and suppose  $M(0) = 0$ ,  $M(x) > 0$  for  $x > 0$ . Considered are two systems: (1)  $l[y] = \lambda y$  ( $0 \leq x < B$ ),  $y(0) = 1$ ,  $y'(0) = h$ , where  $h$  is a real constant,  $Q$  is a real-valued function satisfying  $\int_0^B |dQ(x)| < \infty$  for any  $c$  in  $(0, B)$ ;

(2)  $l[y] = \lambda y$  ( $0 \leq x < B$ ),  $y(0) = 0$ ,  $y'(0) = 1$ ,

where  $Q$  is a real-valued function such that  $\int_0^B x |dQ(x)| < \infty$  for any  $c$  in  $(0, B)$ . The notation is the same as in the article reviewed above. There exists at least one spectral function for each of these two systems. The author gives results connecting the order of growth of the spectral functions as  $\lambda \rightarrow +\infty$  with the behavior of the function  $M$  in the neighborhood of the origin. Theorem I. Let  $\tau$  be any spectral function for (1), and  $b$  any small number in  $(0, B)$ . Then the exact lower bound of the  $\alpha$ , for which each of the following are valid,

$$(i) \int_0^b \left( \int_0^x M(s) ds \right)^{\alpha-1} dx < \infty, \quad (ii) \int_{-\infty}^{\infty} \frac{d\tau(\lambda)}{1+|\lambda|^\alpha} < \infty,$$

$$(iii) \int_0^b [xM(x)]^{\alpha-1} dx < \infty, \quad (iv) \lim_{x \rightarrow 0} x^\alpha M^{\alpha-1}(x) = 0,$$

coincides. If  $\beta$  is this lower bound, and  $\beta > 0$ , and if one of these relations is not valid for  $\alpha = \beta$ , then the preceding relation is not valid for  $\alpha = \beta$ . Theorem II. Let  $\tau$  be any spectral function for (2), and  $b$  as above. The exact lower bound of the  $\alpha$ , for which each of the following relations hold,

$$(a) \int_{-\infty}^{\infty} \frac{d\tau(\lambda)}{1+|\lambda|^\alpha} < \infty, \quad (b) \int_0^b M^{\alpha-1}(x) d(-x^{\alpha-2}) < \infty,$$

$$(c) \lim_{x \rightarrow 0} x^{\alpha-2} M^{\alpha-1}(x) = 0,$$

coincides. If  $\beta$  is the lower bound, and if (a) is valid for  $\alpha = \beta$ , then (b) and (c) are valid for  $\alpha = \beta$ . Theorem III. Let  $M$  be continuous in a right neighborhood of the origin, and suppose

$$\lim_{x \rightarrow 0} x^{-\beta} M^+(x) = N \quad (\beta > -1),$$

where  $M^+(x)$  is the right lower derivative number of  $M$  at  $x$ . If  $\tau$  is any spectral function of (1), then, as  $\lambda \rightarrow +\infty$ ,

$$\tau(\lambda) = N^{-1/(\beta+2)} T(\beta) \lambda^\gamma + o(\lambda^\gamma),$$

where  $T(\beta) = (\beta+2)^{-2\gamma} (\beta+1) \Gamma^{-2}(\gamma+1)$ ,  $\gamma = (\beta+1)(\beta+2)^{-1}$ , and  $\Gamma$  is the gamma-function. If  $\tau$  is any spectral function of (2), then, as  $\lambda \rightarrow +\infty$ ,

$$\tau(\lambda) = N^{1/(\beta+2)} T_1(\beta) \lambda^\delta + o(\lambda^\delta),$$

where

$$T_1(\beta) = (\beta+2)^{-2/(\beta+2)} (\beta+3)^{-1} \Gamma^{-2}(\delta), \quad \delta = (\beta+3)(\beta+2)^{-1}.$$

The last result extends the validity of the asymptotic relations given by V. A. Marčenko [Trudy Moskov. Mat. Obšč. 1 (1952); 327-420; MR 15, 315].

E. A. Coddington (Los Angeles, Calif.).

**Hasse, Maria. Über eine Hillsche Differentialgleichung.** Wiss. Z. Friedrich-Schiller-Univ. Jena/Thüringen. Math.-Nat. Reihe 5 (1955/56), 233-236.

The solution of the Hill differential equation

$$\phi'' + \{\lambda + \cos t/(1 + \varepsilon \cos t)\} \phi = 0$$

is obtained in the form  $c_1 e^{\mu t}/t + c_2 e^{-\mu t} g(t)$ . The quantities  $e^{2\pi\mu}$  and  $e^{-2\pi\mu}$  are evaluated as the roots of a quadratic equation and consequently a diagram in rectangular coordinates  $(\lambda, \varepsilon)$  of the stable solution is constructed. The solution is unstable when the real part of  $\mu$  does not vanish; it is stable when  $\mu$  is pure imaginary, but  $e^{2\pi\mu} \neq \pm 1$ .

S. Kulik (Columbia, S.C.).

**Persidskii, K. P. Infinite systems of differential equations.** Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. 1956, no. 4(8), 3-11. (Russian)

It appears that a few years ago Persidskii was transferred to Kazakhstan (Turkestan; Alma Ata?). He has succeeded in forming there a whole school of young Kazakh (Turcoman) mathematicians whose energies he turned to the very worth-while problem of extending to systems of differential equations in any cardinal number of variables the results known for a finite number of variables. Their work is published in the Proceedings (Izvestia) of the Kazakh Academy of Sciences.

In the present paper the author develops certain general considerations on equations

$$\dot{x} = f(t, x),$$

where  $x$  is a point of a space with an arbitrary (even transfinite) set of coordinates. That is to say  $x$  is a point on the cartesian product of a completely arbitrary collection of real lines. Various examples are discussed.

S. Lefschetz (Princeton, N.J.).

See also: Schwartz, p. 288; Barocia, p. 289; Gates, p. 321; Dalcher, p. 343; Wittmeyer, p. 350; Havlicek, p. 365.

### Partial Differential Equations

**Kasuga, Takashi. Supplement to my paper "On the homogeneous linear partial differential equation of the first order"** Osaka Math. J. 8 (1956), 139-143.

The paper referred to in the title [Osaka Math. J. 7 (1955), 39-67; MR 17, 40] is supplemented by investigating quasi-solutions of partial differential equations of the form

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^n f_\mu(x, y_1, \dots, y_n) \frac{\partial z}{\partial y_\mu} = h(x, y_1, \dots, y_n) + k(x, y_1, \dots, y_n)$$

while formerly only the homogeneous case ( $h = k = 0$ ) was considered.

E. H. Rothe (Ann Arbor, Mich.).

**Danilyuk, I. I. Function theoretical method in the theory of second order differential equations on a surface.** Dopovidi Akad. Nauk Ukrain. RSR. 1956, 423-425. (Ukrainian. Russian summary)

A functional-theoretical method is developed for the theory of elliptic, non-selfadjoint, differential equations of the second order. Applications are given to the first and second boundary value problems on surfaces. This work is an extension of one of the author's earlier works

[Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 11-13; MR 17, 741].  
H. P. Thielman (Ames, Ia.).

**Hartman, Philip.** On the local behavior of solutions of  $\Delta u = g(x, u, \nabla u)$ . Comm. Pure Appl. Math. 9 (1956), 435-445.

This paper is a summary of a previous paper of Hartman and Wintner [Amer. J. Math. 77 (1955), 453-474; MR 17, 855].  
A. Douglis (New York, N.Y.).

**Ėidus, D. M.** On the existence of the normal derivative of the solution of the Dirichlet problem. Vestnik Leningrad. Univ. 11 (1956), no. 13, 147-150. (Russian)

Let  $\Omega$  be a finite region in  $n$ -dimensional space bounded by a surface  $S$  having a twice continuously differentiable parametrization. The author denotes by  $D(S)$  the totality of functions continuously differentiable on  $S$  and defines by

$$\|\varphi\|^2 = \int_S (\varphi^2 + \text{grad}_S^2 \varphi) dS,$$

where  $\text{grad}_S \varphi$  is the surface gradient of  $\varphi$ , the norm in  $D(S)$ . Let  $W_2^{(1)}(S)$  denote the space obtained by the closure of the linear manifold  $D(S)$  with respect to the norm  $\|\varphi\|$ . It is shown that, if the function  $\psi \in W_2^{(1)}(S)$ , then the harmonic function  $u(x)$  which coincides with  $\psi$  on  $S$  has the property that its normal derivative  $\partial u / \partial n$  exists almost everywhere on  $S$ , and belongs to  $L_2(S)$ .

A. J. Lohwater (Ann Arbor, Mich.).

**Pini, Bruno.** Sul comportamento alla frontiera delle derivate delle soluzioni dei problemi fondamentali armonico e biarmonico in due variabili. Rend. Sem. Fac. Sci. Univ. Cagliari 26 (1956), 7-29.

Let  $D$  be a plane domain bounded by a curve  $C$  with the parametric representation  $x=x(s)$ ,  $y=y(s)$ ,  $0 \leq s \leq L$ , where the functions  $x'(s)$ ,  $y'(s)$  satisfy uniform Lipschitz conditions. For  $t$  small the curves  $C_t$ :  $x=x(s, t)=x(s)-ty'(s)$ ,  $y=y(s, t)=y(s)+tx'(s)$  are interior to the curve  $C$ . Let  $u$  be a solution of the problem

$$(1) \quad \Delta u = 0, \text{ in } D; \quad u = f \text{ on } C.$$

The normal and tangential derivatives of  $u$  to the curve  $C_t$  at the point  $(\sigma, t)$  will be denoted respectively by  $u_1 = \partial u(\sigma, t) / \partial t$  and  $u_2 = \partial u(\sigma, t) / \partial \sigma$ . The subarcs of  $C_t$  for which the arc length  $s$  has the range of values

$$0 < \varepsilon < \alpha \leq s \leq \beta < L - \varepsilon$$

will be denoted by  $\Gamma_t$ . If there is an integrable function  $f_t(\sigma)$  such that

$$(2) \quad \lim_{t \rightarrow 0+} \int_{\Gamma_t} |u_1(\sigma) - f_t(\sigma)| d\sigma = 0,$$

then  $u_1$  is said to converge in the mean of order one on the arcs  $\Gamma_t$ . In the present paper it is proved that if the boundary value function  $f(s)$  appearing in (1) is absolutely continuous and the function  $|f'(s+\omega) - f'(s-\omega)|/|\omega|$  is summable on the rectangle  $0 \leq \omega \leq \varepsilon$ ,  $\alpha \leq s \leq \beta$ , then  $u_1$  and  $u_2$  converge in the mean of order one on the arcs  $\Gamma_t$ . These derivatives converge at almost every point on the arc  $\Gamma = \lim_{t \rightarrow 0} \Gamma_t$  when the limits are taken along the normals to  $\Gamma$ . Similar results are shown to hold for the second derivatives of a solution of the boundary value problem

$$\Delta \Delta u = 0, \text{ in } D; \quad \frac{\partial u}{\partial t} = f_1, \quad \frac{\partial u}{\partial s} = f_2 \text{ on } C.$$

F. G. Dressel (Durham, N.C.).

**Levitan, B. M.** On the solution of the problem of Cauchy for the equation

$$\Delta u - q(x_1, x_2, \dots, x_n)u = \frac{\partial^2 u}{\partial t^2}$$

according to the method of Sobolev. Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 337-376. (Russian)

The author considers the Cauchy problem  $\Delta u - qu = u_{tt}$ ,  $u(t=0)=0$ ,  $u_t(t=0)=f$ , where  $q$  and  $f$  are certain functions defined in Euclidean  $n$ -space  $E_n$ ,  $n \geq 3$ . For the case  $n=3$  see the author's recent paper [Trudy Moskov. Mat. Obšč. 4 (1955), 237-290; MR 17, 372]. The principal results are the following. I. Let  $u(x, t) = u(x_1, \dots, x_{2k+1}, t)$  be a solution of the Cauchy problem for odd  $n=2k+1$ ,  $k \geq 1$ . Let

$$v_k(x, t) = \frac{1}{(k-1)!} \int_0^t (t-\tau)^{k-1} u(x, \tau) d\tau,$$

and let  $z_k^{(0)}$  be  $v_k$  for the case  $q(x)=0$ . Put  $v_k = z_k^{(0)} + z_k^{(1)}$ . If in a neighborhood of the point  $x$  the function  $f$  is bounded, and  $q$  has bounded partial derivatives up to order  $n-3$  inclusive, then for small  $t$ ,  $\partial z_k^{(1)} / \partial t = O(t^{k+1})$ . II. Suppose, in a neighborhood of  $x$ , the functions  $f$  and  $q$  have bounded partial derivatives up to orders  $\lambda-1$  and  $n-3$  (inclusive), respectively. Then  $z_k^{(0)}$  and  $z_k^{(1)}$  have, for small  $t$ , bounded partial derivatives with respect to  $t$  up to order  $\lambda$  inclusive, and  $\partial^2 z_k^{(1)} / \partial t^2 = O(t^{k+2-\lambda})$ . III. In a neighborhood of  $x$  let  $q$  have bounded partial derivatives up to order  $2n-7$  inclusive. Then the solution to the Cauchy problem can be represented for small  $t$  in the form

$$u(x, t) = u_0(x, t) + \frac{\partial^{2k-2}}{\partial t^{2k-2}} \int_{|x-y| \leq t} z(x, y, t) / (y) dy,$$

where  $u_0$  is the solution of the Cauchy problem for  $q(x)=0$ , and

$$z(x, y, t) = \sum \alpha(x, \theta_1, \dots, \theta_{2k}) t^{\lambda-\mu} (\ln t)^{\nu} (\ln r)^{\sigma} + O(1),$$

where  $0 \leq \mu \leq 2k-4$ ,  $\lambda \geq \mu+1$ ,  $\nu \geq 0$ ,  $\sigma \geq 0$ , and the  $\alpha$  are bounded functions of  $x$  and the polar angles  $\theta_i$  computed for the radius vector from  $x$  to  $y$ .

The same representation is valid in case  $n$  is even if  $q$  has bounded partial derivatives up to order  $2n-5$  inclusive.  
E. A. Coddington (Los Angeles, Calif.).

**Levitan, B. M.** On expansions according to eigenfunctions of the equation

$$\Delta u + \{\lambda - q(x_1, x_2, \dots, x_n)\}u = 0.$$

Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 437-468. (Russian)

Let  $D$  be a finite simply connected domain in  $E_n$  ( $n \geq 3$ ), and  $B$  its boundary. The eigenvalue problem  $-\Delta u + qu = \lambda u$ ,  $\partial u / \partial n = 0$  on  $B$ , is considered. It is assumed that  $q$  is real-valued on  $D+B$  and has bounded partial derivatives up to order  $2n-7$  inclusive, if  $n$  is odd, and up to order  $2n-5$  inclusive, if  $n$  is even. The results of this paper were announced earlier [Dokl. Akad. Nauk SSSR (N.S.) 97 (1954); 961-964; MR 16, 482], but the author notes that the differentiability assumptions on  $q$  have been modified. The results depend on those presented in the paper reviewed above. Since  $q$  is bounded on  $D+B$  it can be assumed that the spectrum of the eigenvalue problem is non-negative. Let  $\{\omega_m\}$  ( $m=1, 2, \dots$ ) be an orthonormal set of eigenfunctions, with corresponding eigenvalues  $\{\mu_m^2\}$  ( $m=1, 2, \dots$ ). Let  $D_\eta$ , for  $\eta > 0$ , be the set of points in  $D$  whose distance from  $B$  are greater than or

equal to  $\eta$ . There exists a constant  $C_\eta$  such that for  $x, y \in D_\eta$

$$\left| \sum_{\mu < \mu} \omega_m(x) \omega_m(y) - \left( \frac{\mu}{2\pi r} \right)^{1n} I_{1n}(\mu r) \right| < C_\eta \mu^{n-1},$$

where  $r = |x - y|$ , and presumably  $I$  is a Bessel function of the first kind. Let  $f \in L_2(D)$ ,  $n = 2k + 1$ , and put  $c_m = \int_D f \omega_m dx$ ,

$$R_s(x, \mu) = \frac{1}{\Gamma(s+1)} \sum_{\mu < \mu} \left( 1 - \frac{\mu_m^2}{\mu^2} \right)^s c_m \omega_m(x),$$

$$R_s^*(x, \mu) = \int_D \theta_s^*(x, y; \mu) f(y) dy,$$

where

$$\theta_s^*(x, y; \mu) = (2\pi)^{-1n} 2^s \mu^{1n-s} I_{1n-s}(\mu r).$$

If  $f$  is bounded in a neighborhood of  $x$ , and  $s = \frac{1}{2}(n-1)$ , the author proves that  $R_s(x, \mu) - R_s^*(x, \mu) \rightarrow 0$  as  $\mu \rightarrow \infty$ , and this relation is valid uniformly over any interior domain provided  $f$  is bounded on  $D$ .

Now suppose  $n$  is odd,  $n = 2k + 1$ , and assume  $f \in C^{2j}$  on  $D + B$ , and the functions  $f, Lf, \dots, L^{j-1}f$  ( $Lf = (\Delta - q)f$ ) all satisfy the boundary condition  $\partial u / \partial n = 0$  on  $B$ . The function  $f$  is extended to all of  $E_n$  so that it and its partial derivatives up to order  $2j$  inclusive are in  $L_2(E_n)$ . If  $l = k - 2j$ , then  $R_l(x, \mu) - R_l^*(x, \mu) \rightarrow 0$ , and

$$\Gamma(l+1) R_l(x, \mu) \rightarrow f(x),$$

as  $\mu \rightarrow \infty$ , uniformly on each domain inside  $D$ . In particular, if  $k$  is even and  $2j = k$ , then  $\sum_{\mu < \mu} c_m \omega_m(x) \rightarrow f(x)$ ,  $\mu \rightarrow \infty$ .  
E. A. Coddington (Los Angeles, Calif.).

**Levitan, B. M.** On the expansion according to the eigenfunctions of a self-adjoint partial differential equation. *Trudy Moskov. Mat. Obsč.* 5 (1956), 269-298. (Russian)

The author considers the elliptic operator  $L$  given by

$$Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) + cu,$$

where  $a_{ij} (= a_{ji})$ ,  $c$  are analytic on a finite domain  $D$  of  $E_n$ . The eigenvalue problem  $Lu = \lambda u$ ,  $u = 0$  on the boundary  $B$  of  $D$ , is studied, with particular emphasis on the asymptotic behavior of the spectral function, and summability questions concerned with the expansion by eigenfunctions. The methods and results are analogous to those given by the author for the equation  $-\Delta u + qu = \lambda u$ , which appeared in the two papers reviewed above.

E. A. Coddington (Los Angeles, Calif.).

★ **Diaz, J. B.; and Ludford, G. S. S.** On the Euler-Poisson-Darboux equation, integral operators, and the method of descent. Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 73-89. University of Maryland Book Store, College Park, Md., 1956.

This paper consists of three independent sections. In the first of these the singular initial value problem for the Euler-Poisson-Darboux equation

$$\Delta_m u = \frac{\partial^2 u}{\partial t^2} + \frac{k}{t} \frac{\partial u}{\partial t}$$

is solved for  $-\infty < t < \infty$  and  $k > m - 1$  by employing some ideas of Le Roux [*J. Math. Pures Appl.* (5) 6 (1900), 387-422] and Bureau [*Comm. Pure Appl. Math.* 8 (1955), 143-202; MR 16, 826]. Here  $m$  is the dimension of an  $x$ -space and  $\Delta_m$  the Laplace operator. The solution is ex-

tended to other values of  $k$  by a recursion method due to Weinstein [*Proc. Symposia Appl. Math.*, v. 5, McGraw-Hill, New York, 1954, pp. 137-147; MR 16, 137].

In the second section the method of Le Roux is used to give an alternative derivation of the integral operator of the first kind, which was introduced by Bergman [see, e.g., *Amer. J. Math.* 70 (1948), 856-891; MR 10, 752] for the purpose of studying the behavior of solutions of certain hyperbolic equations.

The third section gives a reflection principle for a solution of the Helmholtz equation  $\Delta_m u + \lambda y = 0$  when a linear combination of  $u$  and a directional derivative approaches zero on a plane.  
H. F. Weinberger.

**Babukov, A. G.** On a certain boundary problem of the theory of the deep well pump. *Dokl. Akad. Nauk SSSR* (N.S.) 108 (1956), 39-42. (Russian)

The author considers the simultaneous equations  $\partial^2 u_j / \partial t^2 + \eta \partial u_j / \partial t = a^2 \partial^2 u_j / \partial x^2$ ,  $\eta = \text{const} > 0$ ,  $0 \leq x \leq l$ ,  $j = 1, 2$ , with the conditions  $u_j(x, t + T) = u_j(x, t)$ ,  $u_1(0, t) = f(t)$ ,  $u_2(0, t) = 0$ , and more involved conditions for  $x = l$ , extending his formal method for the case of one unknown function [same *Dokl. (N.S.)* 88 (1953), 635-637; MR 14, 1091].  
F. V. Atkinson (Canberra).

★ **Leray, Jean.** Hyperbolic differential equations. The Institute for Advanced Study, Princeton, N. J., 1953. Reprinted November, 1955. 238 pp.

The 1953 set of these mimeographed notes was reviewed in MR 16, 139.

**Lebedev, V. I.** On a system of parabolic equations. *Dokl. Akad. Nauk SSSR* (N.S.) 103 (1955), 763-766. (Russian)

The author establishes the uniqueness of the solution of the system of parabolic equations

$$(1) \quad \frac{\partial u_k}{\partial t} - a_k \Delta u_k = f_k(X, t) \quad (k = 1, 2, \dots, m),$$

where  $a_k = \alpha_k + i\beta_k$ ,  $\alpha_k > 0$ ,  $X = (x_1, x_2, \dots, x_n)$  in the domain  $Q = \Omega \times [0, l]$  where  $\Omega$  is a bounded piecewise smooth surface  $S$ :  $f_k \in L_2(Q)$ . The boundary conditions are

$$(2) \quad u_k \Big|_{t=0} = \phi_k(X) \in W_2^{(1)}(\Omega), \quad \frac{\partial u_k}{\partial n} \Big|_S = \sum_{i=1}^m b_{ki}(X) u_i \Big|_S,$$

where the  $b_{ki}$  are subject to

$$(3) \quad c_k b_{ki}(X) \Big|_S = c_i \bar{b}_{ik}(X), \quad \sum_{k,i=1}^m c_k b_{ki} \bar{b}_{ik} \Big|_S \leq 0, \quad c_k > 0.$$

The  $b_{ki}$  are continuous functions and conditions (3) hold for a distance  $h_0$  inside the domain  $\Omega$ .  
C. G. Maple.

**Karol', I. L.** Boundary problems for an equation of mixed elliptic-hyperbolic type. *Dokl. Akad. Nauk SSSR* (N.S.) 101 (1955), 793-796. (Russian)

The author considers a variety of boundary value problems for the equation  $u_{xx} + \gamma u_{yy} + \alpha u_y = 0$  in a region in which (\*) is of mixed elliptic-hyperbolic type. The case  $\alpha > 0$ , a constant, is discussed thus extending the author's previous work [same *Dokl. (N.S.)* 88 (1953), 397-400; MR 14, 757] for the case  $\alpha < 0$ . The domain considered is bounded in  $y > 0$  by arcs  $\{\Gamma_k\}$  ( $k = 0, 1, 2, \dots, n$ ) with endpoints  $P_i(x_i, 0)$  ( $i = 0, 1, 2, \dots, 2n+1$ ) with the property that  $\Gamma_0$  lies "outside" all the  $\Gamma_k$ . In  $y < 0$  the domain is bounded by the characteristics of (\*) through the points  $P_i$ .  
M. H. Protter.

**Guderley, Gottfried.** On the development of solutions of Tricomi's differential equation in the vicinity of the origin. *J. Rational Mech. Anal.* 5 (1956), 747-790.

In terms of the variables  $\rho = -\eta^2 + (3\theta/2)^2$ ,  $\xi = \eta(3\theta/2)^{-2/3}$ , the Tricomi equation,  $\psi_{\eta\eta} - \eta\psi_{\theta\theta} = 0$  has solutions of the form  $\psi = \rho^{n/2}G(\xi)$ . Only those solutions are considered which are odd functions of  $\theta$  and hence vanish for  $\xi = -\infty$ . A first family of particular solutions is obtained by the condition that  $\psi$  and its derivatives be finite on the characteristic  $\xi = 1$ . This condition arises naturally in the physical problem under consideration but it does not seem to lead to a complete system, and it is not at all clear that all physically significant solutions of Tricomi's equation can be represented by superposition of particular solutions of the first family. To overcome this difficulty, the author defines a second family of particular solutions by the condition  $G(c_2) = 0$ , taking at first  $c_2 \neq 1$  and then carrying out the limiting process  $c_2 \rightarrow 1$ . The solutions of the second family do form a complete system, and by means of these solutions the author constructs the solution of a certain inhomogeneous boundary value problem. He then manipulates the solution thus obtained analytically (by contour integration techniques), transforming it finally into an infinite series of solutions of the first family.

A. Erdélyi (Jerusalem).

**Greco, Donato.** Sul problema di Lauricella per una particolare equazione del quarto ordine. *Boll. Un. Mat. Ital.* (3) 11 (1956), 394-401.

Let the boundary curve  $\Gamma$  of the plane domain  $T$  have the parametric representation  $x_i = \varphi_i(s)$ ,  $i = 1, 2$ . Each of the functions  $\varphi_i(s)$  is assumed to belong to class  $C^{(4,\lambda)}$ , that is each function possesses a fourth derivative which satisfies a Hölder condition. The external normal to  $\Gamma$  will be indicated by the letter  $n$ . It is assumed that the fourth order differential operator  $L = M\Delta$  can not be reduced to the product  $\Delta\Delta$  where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, M = a_{11} \frac{\partial^2}{\partial x_1^2} + 2a_{12} \frac{\partial^2}{\partial x_1 \partial x_2} + a_{22} \frac{\partial^2}{\partial x_2^2}.$$

Here  $a_{11}a_{22} - a_{12}^2 = 1$  and the  $a_{ij}$ 's are constants. Let  $f_1(x)$ ,  $f_2(x)$  be given function defined on  $\Gamma$  such that  $f_1$  belongs to class  $C^{(1,\lambda)}$  and  $f_2(x)$  to class  $C^{(0,\lambda)}$ . The author states that the problem of Lauricella consists of determining a function  $u$  of class  $C^{(4)}$  in  $T$  and of class  $C'$  in  $T + \Gamma$  which satisfies the equation  $Lu = 0$  in  $T$  and the conditions  $u = f_1$ ,  $du/dn = f_2$  on the curve  $\Gamma$ . The author proves that the problem of Lauricella has a unique solution  $u$  representable in the form  $u = u_1 + u_2$ , where  $\Delta u_1 = 0$  and  $Mu_2 = 0$  in  $T$ .

F. G. Dressel.

**Pini, Bruno.** Su una generalizzazione del problema fondamentale di valori al contorno per l'equazione del calore iterata. *Rend. Sem. Fac. Sci. Univ. Cagliari* 26 (1956), 30-57.

Let  $\chi_1(y) < \chi_2(y)$  be two functions which together with their first derivatives are continuous on  $0 \leq y \leq 1$ . The set of points  $(x, y)$  defined by the inequalities  $0 < y < 1$ ,  $\chi_1(y) < x < \chi_2(y)$  will be denoted by the letter  $D$ . Let  $f_i(y)$ ,  $\varphi_i(y)$ ,  $i = 1, 2$ , be four given functions defined on  $0 \leq y \leq 1$ , the  $f_i$  ( $f_i(0) = 0$ ) being absolutely continuous and the  $\varphi_i$  summable there. The main object of this paper is to show that the following boundary value problem has one and only one solution: Determine a function  $u = u(x, y)$

such that

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}\right)^2 u = 0, \text{ in } D,$$

$$u = 0, \frac{\partial u}{\partial y} = 0, \text{ on } y = 0, \chi_1(0) < x < \chi_2(0),$$

$$\lim_{t \rightarrow 0+} \int_0^1 \left( |(u)_{\gamma_i(t)} - f_i| + \left| \left( \frac{\partial u}{\partial x} \right)_{\gamma_i(t)} - \varphi_i \right| \right) dy = 0, i = 1, 2.$$

Here  $\gamma_i(t)$  indicates the arc given by the equation  $x = \chi_i(y) - (-1)^i t$ ,  $0 \leq y \leq 1$ .

F. G. Dressel.

**Montaldo, Oscar.** Su un problema di valori al contorno per le funzioni bicaloriche. *Rend. Sem. Fac. Sci. Univ. Cagliari* 26 (1956), 1-6.

Let  $\chi_1(y) < \chi_2(y)$  be two functions that belong to class  $C'$ , and denote by  $D$  the plane domain  $0 < y < a$ ,  $\chi_1(y) < x < \chi_2(y)$ . The arcs  $x = \chi_i(y)$ ,  $0 \leq y \leq a$ , will be denoted by  $\gamma_i$ , where  $i = 1, 2$ . The letter  $C$  will stand for the characteristic  $y = 0$ ,  $\chi_1(0) \leq x \leq \chi_2(0)$  of the equation

$$(*) \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}\right)^2 u = 0.$$

In the present paper the author treats the problem of finding a solution  $u = u(x, y)$  of equation (\*) in  $D$  and satisfying the following boundary conditions:

$$u = f_i(y), \lambda_i(y)u_x + \mu_i(y)u_{xx} = \varphi_i(y), \text{ on } \gamma_i, i = 1, 2,$$

$$u = f(x), \lambda(x)u_y + \mu(x)u_{yy} = \varphi(x), \text{ on } C.$$

The above boundary value problem is reduced to a system of integral equations and then shown to have a solution if the functions  $\lambda(x)$ ,  $\mu(x) > 0$ ,  $\lambda_i(y)$ ,  $\mu_i(y) > 0$ ,  $\varphi(x)$ ,  $\varphi_i(y)$ ,  $f''(x)$ ,  $f_i''(y)$  are continuous.

F. G. Dressel (Durham, N.C.).

**Martynov, G. A.** On the solution of the inverse problem of Stephan for the semi-space when the phase boundary moves according to a linear law. *Dokl. Akad. Nauk SSSR (N.S.)* 109 (1956), 279-282. (Russian)

The problem considered is to find functions  $\theta_1(x, t) \leq 0$ ,  $\theta_2(x, t) \geq 0$  which satisfy the conditions:

$$\frac{\partial \theta_1}{\partial t} = a_1 \frac{\partial^2 \theta_1}{\partial x^2} \quad (0 \leq x \leq h(t)),$$

$$\frac{\partial \theta_2}{\partial t} = a_2 \frac{\partial^2 \theta_2}{\partial x^2} \quad (h(t) \leq x \leq \infty; h(0) = 0),$$

$$\theta_2(x, 0) = f(x) \geq 0,$$

$$\lambda_1 \frac{\partial \theta_1(h, t)}{\partial x} - \lambda_2 \frac{\partial \theta_2(h, t)}{\partial x} = q \frac{dh}{dt}.$$

The author discusses, by approximate methods, the case  $h = vt$ .

R. Finn (Pasadena, Calif.).

**Bahvalov, N. S.** Conditions for convergence and the order of error for a solution of Cauchy's problem for a linear first-order equation by the method of finite differences. *Prikl. Mat. Meh.* 20 (1956), 279-283. (Russian)

The author approaches the partial differential equation

$$(1) \quad \partial u / \partial t + a(t, x) (\partial u / \partial x) - b(t, x) u = 0$$

with Cauchy data

$$u|_{t=0} = u_0(x)$$

by means of the difference equation

$$(2) \quad L_{nm}(v_{nm}) = (v_{n+1,m} - v_{nm})/h + a_{nm}(v_{nm} - v_{n,m-1})/l - b_{nm}v_{nm} = 0, \\ v_{0m} = u_0(ml),$$

where  $f_{nm} = f(t, x)$  with  $t = nh$ ,  $x = ml$ . By a change of independent variables in (1) one may achieve  $a(t, x) \geq 0$ . Courant, Friedrichs, and Lewy [Math. Ann. 100 (1928), 32-74] have proved that, for smooth coefficients  $a$  and  $b$ , solutions of (2) converge to solutions of (1). The present paper drops the assumption of smoothness and considers generalized solutions in the sense of Sobolev [cf. Petrovsky, Lectures on partial differential equations, Gostehizdat, Moscow, 1950, p. 84; MR 13, 241; 16, 478]. It is proved that, under suitable hypotheses, solutions of (2) again converge to those of (1), and the order of the error is estimated. It is necessary that  $h$  and  $l$  satisfy certain relations, among them that  $l \geq a(t, x)h$ . During the course of the proof it is shown that solutions of (2) are stable with respect to round-off error. The proof depends upon the recursion formula

$$v_{n+1,m} = [1 - a_{nm}h/l + b_{nm}h]v_{nm} + (a_{nm}h/l)v_{n,m-1}$$

and on the consequent form for  $v_{p,q}$ , of fixed  $s$ ,

$$v_{pq} = \sum_{r=-\infty}^{\infty} I_{sr} v_{qr}$$

involving a finite number of non-zero coefficients  $I_{sr} v_{qr}$ . Two devices from probability are employed: Chebyshev's inequality, and a relation between the binomial and normal distributions. The method follows somewhat that of S. N. Bernstein [Sobšč. Har'kov. Mat. Obsč. (2) 13 (1912), 1-2=Collected works, v. I, Izdat. Akad. Nauk SSSR, Moscow, 1952, pp. 105-106; MR 14, 2], who dealt with the case where  $a$  is a constant and  $b$  is zero.

R. B. Davis (Syracuse, N.Y.).

**Avakumović, Vojislav G.** Über die Eigenwerte der Schwingungsgleichung. Math. Scand. 4 (1956), 161-173.

Let  $G(P, Q, \lambda)$  be the Green's function of the Helmholtz operator on a finite domain  $D$ . Let  $l_P$  be the distance from the boundary of a point  $P$ ,  $r_{PQ}$  the distance from  $P$  to  $Q$ ,  $K_0(x)$  a modified Bessel function, and  $\theta = \arg \lambda$  with  $0 < \theta < 2\pi$ . The author shows that there exists a constant  $m$  independent of  $Q$  and  $\lambda$  such that

$$|G(P, Q; \lambda) - (2\pi)^{-1} K_0(r_{PQ}(-\lambda)^{1/2})| \leq m l_P^{-3/2} (\sin \frac{1}{2}\theta)^{-1} |\lambda|^{-1/4} \exp(-\frac{1}{4} l \operatorname{Re}[( -\lambda)^{1/2}]).$$

(There is a misprint in the exponent of  $\sin \frac{1}{2}\theta$  in the statement of the theorem.)

Using this estimate, the author establishes a necessary and sufficient condition for the Fourier series of a function  $f(P)$  in terms of the membrane eigenfunctions  $\Phi_\nu(P)$  of the domain  $D$  to be summable to  $f$  in the sense that

$$\lim_{\sigma \rightarrow \infty} \sum_{\lambda_\nu \leq \sigma} \{1 - \exp(\lambda_\nu - \sigma)\}^\sigma \alpha_\nu \Phi_\nu(P) = f(P).$$

Here the  $\lambda_\nu$  are the membrane eigenvalues of  $D$ , and  $\sigma$  and  $\alpha$  are constants.

H. F. Weinberger.

**\*Pleijel, Åke.** On the eigenfunctions of the membrane equation in a singular case. Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 197-207. University of Maryland Book Store, College Park, Md., 1956.

The three-dimensional eigenvalue problem

$$(1) \quad \Delta u + \lambda h(x, y, z)u = 0$$

in a finite domain  $V$  with  $u=0$  on the boundary is considered. The author has previously shown [Amer. J. Math. 70 (1948), 892-907; MR 10, 301] that if the eigenvalues are  $\lambda_1 \leq \lambda_2 \leq \dots$  and the corresponding eigenfunctions at a point  $p$  have the values  $\varphi_1(p), \varphi_2(p), \dots$ , then

$$(2) \quad \sum_{\lambda_n < \lambda} \varphi_n^2(p) = \frac{k(p)^{1/2} \lambda^{3/2}}{6\pi^2} + o(\lambda^{3/2}).$$

The present paper is concerned with the case where  $k \geq 0$  and where  $p$  lies in an interior subdomain  $V_0$  of  $V$  in which  $k=0$ . In this case, the leading term of (2) vanishes.

It is shown that for  $p$  and  $q$  in  $V_0$  the limit as  $\lambda \rightarrow -\infty$  of the Green's function  $G(p, q; \lambda)$  of (1) for the domain  $V$  is the value  $g(p, q)$  of the Green's function of  $\Delta u=0$  for the domain  $V_0$ . From this, together with the Fourier development of  $G(p, q; \lambda) - g(p, q)$  follows the relation

$$(3) \quad \sum_{\lambda_n < \lambda} \varphi_n(p) \varphi_n(q) = O(\lambda)$$

for  $p$  and  $q$  in  $V_0$ .

For the particular case when  $k$  is a positive constant  $c^2$  outside  $V_0$ , the author finds the stronger result

$$(4) \quad \sum_{\lambda_n < \lambda} \varphi_n(p) \varphi_n(q) = 2\lambda^{1/2} \pi^{-1} c^{-1} \iint_{S_0} \frac{\partial g}{\partial \nu}(p, \cdot) \frac{\partial g}{\partial \nu}(q, \cdot) dS_0 + o(\lambda)^{1/2},$$

where  $S_0$  is the boundary of  $V_0$  and  $p$  and  $q$  are in  $V_0$ .

H. F. Weinberger (College Park, Md.).

**Levitan, B. M.** On differentiation of the spectral function of the Laplace operator. Mat. Sb. N.S. 39(81) (1956), 37-50. (Russian)

Let  $D$  be a smoothly bounded open subset of real  $n$ -space and  $A$  the self-adjoint operator on  $L^2(D)$  determined by Laplace's operator and the Neumann boundary condition. Let  $A = \int \lambda dE_\lambda$  be the spectral resolution of  $A$ ; the projection  $E_\lambda$  is given by a kernel  $e(\lambda, x, y)$  called the spectral function of  $A$ . Let

$$e_0(\lambda, x, y) = (2\pi)^{-n} \int_{|\xi| \leq \lambda} e^{i\xi(x-y)} d\xi \quad (|\xi|^2 = \sum \xi_k^2),$$

be the spectral function corresponding to the entire space. The author has shown that  $e(\lambda, x, y) = e_0(\lambda, x, y) + O(\lambda^{1/(n-1)})$  uniformly on compact subsets of  $D \times D$ . Now he proves that this estimate can be differentiated with respect to  $x$  and  $y$  any number of times provided that the error term is multiplied by  $\lambda^{k/2}$ , where  $k$  is the order of the differentiation. An analogous formula holds for the Riesz means. {Reviewer's remark. A corresponding estimate is probably true when  $D$  is not necessarily bounded and when  $A$  is a semibounded selfadjoint of any elliptic differential operator with constant coefficients. See Gårding, Kungl. Fysiogr. Sällsk. i Lund Förh. 24 (1954), no. 21; MR 17, 158.}

L. Gårding (Lund).

**Pleijel, Åke.** Remarks on Courant's nodal line theorem. Comm. Pure Appl. Math. 9 (1956), 543-550.

Let  $V$  be a connected domain with boundary  $S$  in the  $xy$ -plane. Eigenvalue problems of the following type are considered:

$$(1) \quad \begin{aligned} \Delta u + \zeta u &= 0 \text{ in } V, \\ u &= 0 \text{ on } S_0 \subset S, \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } S_1 = S - S_0, \end{aligned}$$

where  $\Delta$  is the Laplace operator and  $\nu$  the normal on  $S$ . In the fixed membrane problem ( $S_1=0$ ) the author sharpens the Courant nodal line theorem by proving that only for a finite number of eigenvalues can the  $n$ th eigenfunction divide  $V$  into exactly  $n$  nodal domains. By Courant's theorem this is the maximal division by nodal lines. The author suggests that the same result is probably true in the case of the free membrane ( $S_0=0$ ). In the case of the fixed square membrane it is shown that only for  $n=1, 2$ , and  $4$  does the  $n$ th eigenfunction divide the region into exactly  $n$  nodal domains. L. E. Payne.

**Avakumović, Vojislav G.** Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten. *Math. Z.* **65** (1956), 327–344.

Let

$$\frac{1}{g^{\frac{1}{2}}} \frac{\partial}{\partial x_1} \left( g^{\frac{1}{2}} g^{ik} \frac{\partial}{\partial x_k} \right)$$

be the Laplace operator on a compact  $N$ -dimensional Riemannian space  $C^\infty$  of total volume  $L$  and let  $0=\lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ ,  $\Phi_0=1$ ,  $\Phi_1, \Phi_2, \dots$  be its eigenvalues and eigenfunctions. For the dimension  $N=3$  the author proves the precise estimate

$$\sum_{\lambda_n \leq x} 1 = \frac{1}{6\pi^2} x^{3/2} + O(x) = \sum_{\lambda_n \leq x} \Phi_n^2(P),$$

as  $x \rightarrow \infty$  and  $O(x)$  cannot be replaced by  $o(x)$  even for the 3-sphere. The previously best known estimate was only  $O(x^{3/2}) = O(x^{N/2})$ , due to Minakshisundaram and Pleijel [*Canad. J. Math.* **1** (1949), 242–256; MR **11**, 108]. For the ordinary Laplacian in an  $N$ -dimensional domain with sufficiently smooth boundary this precise estimate had been previously established by the author [*Acad. Serbe Sci. Publ. Inst. Math.* **4** (1952), 95–96; MR **14**, 278] and by Levitan [*Dokl. Akad. Nauk SSSR (N.S.)* **90** (1953), 133–135; MR **15**, 129]. S. Bochner (Princeton, N.J.).

See also: Schwartz, p. 288; Lisevič, p. 296; Widder, p. 304; Čudov, p. 309; Birman, p. 316; Harish-Chandra, p. 318; Grothendieck, p. 327; *Memoirs of the unifying study...*, p. 332; Cherry, p. 358; Chambré, p. 358; Dul'nev and Kondrat'ev, p. 359; Chambers, p. 359; Adem, p. 365.

### Difference Equations, Functional Equations

**Zaharčuk, E. Yu.** Solutions of the equation  $f(\varphi(x))=f(x)$ . *Grodzenskii Gos. Ped. Inst. Uč. Zap.* **1** (1955), 35–40. (Russian)

See also: Skolem, p. 275; Bellman, p. 286; Eichler, p. 299; Bahvalov, p. 314.

### Calculus of Variations

**Cesari, Lamberto.** On the calculus of variations in two variables. *Comm. Pure Appl. Math.* **9** (1956), 363–371.

The paper is a useful supplement to an earlier paper [*Amer. J. Math.* **74** (1952), 265–295; MR **14**, 292], a

theorem established also by Sigalov [*Uspehi Mat. Nauk (N.S.)* **6** (1951), no. 2(42), 16–101; MR **13**, 257, 1139; **14**, 769] and by Danskin [*Riv. Mat. Univ. Parma* **3** (1952), 43–63; MR **14**, 292]. The paper consists of a summary of the author's proof, together with an account of the background and references to further related researches.

L. C. Young (Madison, Wis.).

**Auslander, Louis.** Remark on the use of forms in variational calculations. *Pacific J. Math.* **6** (1956), 209–210.

The present note is an addendum to an earlier paper by the same author [same *J.* **5** (1955), 853–859; MR **17**, 862] in which a certain differential form had been introduced. By proving a certain uniqueness theorem the author clearly indicates his reasons for the introduction of this particular form. H. Rund (Durban).

**Young, L. C.** Some new methods in two-dimensional variational problems with special reference to minimal surfaces. *Comm. Pure Appl. Math.* **9** (1956), 625–632.

Generalized surfaces  $(L, f)$  with discontinuous integrands  $f$  are here considered as Banach linear functionals. A real function  $\varphi(x, j)$  of the real 3-vectors  $x, j$  is said to be homologous to 0 if (a)  $\varphi$  is bounded for  $|x| < a$ ,  $|j| = 1$ ; (b)  $\varphi(x, tj) = t\varphi(x, j)$  for all  $t \geq 0$ ; (c) for every linear function  $x(u, v)$ ,  $u, v$  real, with (constant) jacobian  $j \neq 0$  the composite function  $\varphi[x(u, v), j]$  is the derivative of its integral with respect to  $u$  and  $v$  almost everywhere; (d) for every continuous polyhedron  $L(f) = \iint [x, \theta(x)] d\mu$  — thought of as a Banach linear functional — we have  $\int \varphi[x, \theta(x)] d\mu = 0$ . Then any function  $f + \varphi$  is said to be homologous to  $f$  if  $\varphi$  is homologous to 0. The following homology principle is then a consequence of the Hahn-Banach theorem: (P) If  $\lambda_0$  is the boundary of a polyhedron (in the sense of L. C. Young), if  $L_0(f_0) \leq L(f_0)$  for all generalized surfaces  $L(f)$  with boundary  $\lambda_0$ , then there exists a non-negative  $f_0 + \varphi$ , homologous to  $f_0$ , such that  $(L_0, f_0 + \varphi) = 0$ . It is then shown that to every  $\varphi$  homologous to 0 there corresponds a vector function  $\phi(x)$  such that for every Lipschitzian vector  $x(u, v)$  of jacobian  $j(u, v)$ , the relation  $\varphi[x(u, v), j(u, v)] = \phi[x(u, v)] \cdot j(u, v)$ . The vector function  $\phi(x)$  thus associated to  $\varphi$  is termed a field. These concepts are then used to derive properties of discontinuous minimal generalized surfaces.

L. Cesari (Lafayette, Ind.).

**Birman, M. Š.** Variational methods of solution of boundary problems analogous to the method of Trefftz. *Vestnik Leningrad. Univ.* **11** (1956), no. 13, 69–89. (Russian)

The main results of the second half of this article, which gives a variant of Trefftz' method for thin plates, have been announced before [*Dokl. Akad. Nauk SSSR (N.S.)* **101** (1955), 201–204; MR **17**, 41]; the first part gives a similar variant for the mixed boundary value problem for Laplace's equation. L. Gårding (Lund).

See also: Valcovici, p. 346; Kolsrud, p. 347; Ono, p. 348; Toupin, p. 349; Wittmeyer, p. 350; Prager, p. 351; Hill and Power, p. 354; Chandrasekhar, p. 357; Chambers, p. 359.

# TOPOLOGICAL ALGEBRAIC STRUCTURES

## Topological Groups

★Shoenfield, J. R. The structure of locally compact groups. Mathematics Department, Duke University, Durham, North Carolina, 1956. 63 pp. (mimeographed). \$ 1.00.

The author gives a complete systematic exposition on the structure of locally compact groups. The methods used here are, however, strictly the same methods introduced by Gleason, Montgomery, Zippin and the reviewer.

The definition of Lie groups is given similar to that of Chevalley. Using this, the concept of generalized Lie groups is introduced. Then a proof for Kuranishi's extension theorem is given with complete details. Applying Gleason's smoothing process improved by the reviewer, the author shows that small subgroups generate a compact subgroup from which it follows that every locally compact connected group has a compact normal subgroup such that the factor group is a group with no small subgroups.

For proving the last step that a group with no small subgroups is a Lie group, the author follows the line taken by Gleason and by the reviewer. However, careful checks are given at every step.

One can show that the last step is independent of other theorems like Kuranishi's theorem (7.7) and E. Cartan-v. Neumann's theorem (4.9); as a matter of fact these theorems are proved as direct consequences of the theorem that a group with no small subgroups is a Lie group.

H. Yamabe (Minneapolis, Minn.).

Watanabe, Yôiti. A classical theory of the collective description. Progr. Theoret. Phys. 16 (1956), 1-22.

The collective motion of a many body system is treated in a classical way. The approach is that of using auxiliary coordinates whose behavior characterizes the motion of a group of particles. These extra coordinates are introduced by means of continuous point transformations. As the simplest example, the center-of-mass motion is treated first through parameters of the space transformation. It is evident already in this simple case that the general method also supplies conditions of constraint which restrict the relative motion of the individual particles. As another example the two dimensional incompressible and irrotational flow of particles is discussed. This is followed by the general treatment in which the Lagrangian and the Hamiltonian are transformed from the "particle representation" to a representation involving  $r$  coordinates describing  $r$  modes of a certain collective motion. This transformation and the construction of the new Lagrangian and Hamiltonian are discussed in detail. One also gets "gauge conditions" and through them subsidiary conditions. The general treatment is also applied to the vortex motion in hydrodynamics. Further transformations are then applied to bring about a clearer physical picture, changing the subsidiary conditions into a set of stringent constraints ("principle of stiffening"). Two more examples and a group theoretical appendix are also given.

M. J. Moravcsik (Upton, N.Y.).

Gluškov, V. M. Nilpotent product of topological groups. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 3(69), 119-123. (Russian)

The author extends to arbitrary topological groups the

concept of nilpotent product, which was introduced for abstract groups by O. N. Golovin [Mat. Sb. N.S. 27(69) (1950), 427-454; MR 12, 672]. Definition 1. Let  $G$  be an abstract group and  $\{P_\alpha\}$  a family of subsets of  $G$ . The sets  $P_\alpha$  are said to be  $k$ -nilpotently imbedded in  $G$  if for any  $k+2$  elements,  $g_1, \dots, g_{k+2} \in \cup P_\alpha$ , the commutator  $((\dots(g_1, g_2), \dots), g_{k+2})$  is equal to the identity whenever there are two distinct indices  $\alpha, \beta$  such that  $g_1 \in P_\alpha, g_2 \in P_\beta$  (the sets  $P_\alpha, P_\beta$  may coincide). Here  $(a, b) = a^{-1}b^{-1}ab$ . Definition 2. Let  $\{G_\alpha\}$  be a family of topological groups and  $k$  a non-negative integer. The topological group  $G$  is called the free  $k$ th nilpotent product of the groups  $G_\alpha$  if the following three conditions are satisfied: (i) The groups  $\{G_\alpha\}$  are  $k$ -nilpotently imbedded in  $G$ . (ii)  $G$  is topologically generated by the elements of the groups  $G_\alpha$ . (iii) If  $H$  is any topological group for which continuous homomorphisms  $\phi_\alpha: G_\alpha \rightarrow H$  exist for all  $\alpha$ , such that  $\{\phi_\alpha(G_\alpha)\}$  are  $k$ -nilpotently imbedded in  $H$ , then there is a continuous homomorphism  $\phi: G \rightarrow H$ , agreeing with  $\phi_\alpha$  on  $G_\alpha$ .

Theorem. The group  $G$  always exists;  $G_\alpha \cap G_\beta = \{e\}$  for  $\alpha \neq \beta$ ;  $G$  is algebraically generated by the elements of the groups  $G_\alpha$ ; if  $G'$  is any other group satisfying (i), (ii), (iii) then there is a topological isomorphism of  $G$  onto  $G'$  which is the identity automorphism on each of the subgroups  $G_\alpha$ . The author states that the free  $k$ th nilpotent product of compact connected commutative groups coincides for every  $k$  with their free zero nilpotent product, and is therefore an abelian group. On the other hand if one regards them as abstract groups their  $k$ th nilpotent product in the sense of Golovin is not abelian.

Allen Shields (Ann Arbor, Mich.).

Kakar, A. G. On the inner product of S-functions. J. London Math. Soc. 31 (1956), 485-490.

In this paper a method due to D. E. Littlewood [same J. 31 (1956), 89-93; MR 17, 583] is used to check some calculations of inner products of S-functions made by Murnaghan [Amer. J. Math. 60 (1938), 761-784] and to extend his results. G. de B. Robinson (Toronto, Ont.).

Mostert, Paul S.; and Shields, Allen L. On continuous multiplications on the two-sphere. Proc. Amer. Math. Soc. 7 (1956), 942-947.

Let  $S$  be a compact connected manifold (without boundary) and suppose that  $S$  is provided with a continuous associative multiplication. If  $S$  has a unit it is well-known that  $S$  is a group but if it is merely assumed that  $S^2=S$  then no information is available concerning the algebraic structure of  $S$ . It is conjectured in this case that  $S$  is either a group or is simple (contains no proper ideal), this being so if  $S$  is a 1-sphere. If  $S$  is an  $n$ -sphere, if  $S^2=S$  and if  $S$  is not a group then this conjecture is equivalent to the assertion that each element is a right zero or a left zero. In this note it is assumed that  $S$  is a 2-sphere in which the circle-group  $B$  can be imbedded. It follows that  $S$  has a zero. If  $C$  is the complementary domain of  $B$  which contains 0 then  $C^*$  is an ideal in the center of  $S$  and  $Sa=0=aS$  for some  $a \in S \setminus (C^* \cup S^2)$ . Moreover,  $C^*$  is the Rees quotient  $I \times B / \{0\} \times B$  for a suitable multiplication on  $I=[0, 1]$ , 0 and 1 playing their usual roles. (There is some overlap with results obtained independently by the reviewer (in press) which are applicable when  $S$  is the  $n$ -sphere.) The note (which stems

from a question raised by S. T. Hu) also contains interesting examples and some problems. From various remarks it seems likely that the authors at one time intended to assert the existence of a "northernmost" circle-group, 0 being the South Pole.

A. D. Wallace.

**Loewner, Charles.** On some transformation semigroups.

J. Rational Mech. Anal. 5 (1956), 791–804.

Verfasser nennt eine Transformation  $\tau$  der projektiven Gruppe in  $n$  Dimensionen eine projektive Verschiebung, wenn zu  $\tau$  eine projektive Transformation  $\pi$  existiert, so daß  $\tau^* = \pi^{-1}\tau\pi$  eine gewöhnliche Verschiebung darstellt. Eine solche Transformation  $\tau$  läßt dann alle Punkte einer Hyperebene  $l$  durch einen Punkt  $p$  fest und alle Hyper-ebenen durch  $p$ . Solche " $(p, l)$ -Verschiebungen" mit gegebenen  $p$  und  $l$  bilden eine einparametrische Gruppe.

Nun sei  $p$  ein Randpunkt eines offenen konvexen Gebietes  $D$  und  $l$  eine Stützhyperebene des Gebietes  $D$  durch  $p$ . Eine  $(p, l)$ -Transformation transformiert  $D$  entweder in ein Teilgebiet von  $D$  oder in ein Gebiet, welches  $D$  enthält. Transformationen der ersten Art werden projektive Verschiebung von  $D$  genannt; sie bilden mit ihren Grenzfällen die Halbgruppe  $\mathfrak{P}_D$ . Analog nennt Verfasser eine Transformation  $\tau$  der Möbiusgruppe eine Möbiusverschiebung, wenn zu  $\tau$  eine Möbiustransformation  $\mu$  existiert, so daß  $\mu^* = \mu^{-1}\tau\mu$  eine gewöhnliche Verschiebung darstellt. Eine solche Transformation  $\tau$  läßt alle Kreise, die ein gewisses Linienelement  $l$  enthalten fest. Diese " $l$ -Transformationen" bilden bei gegebenen  $l$  eine einparametrische Gruppe.

Nun sei ein offener Bereich des  $n$ -dimensionalen Möbius-Raumes berandet von einer Jordanschen Hyperfläche der Klasse  $C_1$ . Jeder Kreis, der beliebige Linienelemente senkrecht zur Randfläche enthält, soll  $D$  in einem zusammenhängenden Segment schneiden. Dann transformiert eine " $e$ -Transformation"  $D$  entweder in ein Teilgebiet von  $D$  oder in ein Gebiet, welches  $D$  enthält. Transformationen der ersten Art heißen Möbius-Verschiebungen von  $D$ ; sie bilden wiederum zusammen mit ihren Grenzfällen eine Halbgruppe  $\mathfrak{M}_D$ .

Im eindimensionalen Falle fallen Möbius- und projektive Gruppe zusammen. Für  $D$  bestehen jetzt nur zwei Möglichkeiten: entweder ist  $D$  die projektive Gerade, von welcher ein Punkt herausgenommen ist, oder  $D$  ist ein Abschnitt der projektiven Geraden mit zwei Randpunkten. Im ersten Fall kann  $D$  durch die  $x$ -Achse mit dem Punkt  $\infty$  als einzigen Randpunkt gegeben werden, im zweiten durch die Halbgerade  $x > 0$  mit 0 und  $\infty$  als Randpunkten. Im ersten Fall besteht  $\mathfrak{P}_D$  aus allen gewöhnlichen Verschiebungen

$$x' = x + a \quad (-\infty < a < \infty).$$

Im zweiten Fall gehören zu den Randpunkten 0 und  $\infty$  zwei Typen von Transformationen:

$$x' = \frac{x}{1+bx}, \quad x' = x + a, \quad a \geq 0, \quad b \geq 0.$$

Der Zusammensetzungsprozeß der Transformationen dieser beiden Typen entspricht der Bildung von Kettenbrüchen mit positiven Elementen. Daher läuft das Studium der Gruppen  $\mathfrak{P}_D$  und  $\mathfrak{M}_D$  allgemein auf eine Verallgemeinerung der Theorie der Kettenbrüche auf höhere Dimensionen hinaus. — In den weiteren Ausführungen behandelt Verfasser Verzerrungssätze für die Halbgruppen  $\mathfrak{P}_D$  und  $\mathfrak{M}_D$ . Die allgemeinen Ergebnisse werden für  $\mathfrak{M}_D$  mit Beschränkung auf ebene Kreisgebiete weiterhin verfeinert. Schließlich werden gewisse der Halb-

gruppen  $\mathfrak{P}_D$  und  $\mathfrak{M}_D$  rein gruppentheoretisch charakterisiert. Die Untersuchung infinitesimaler Verschiebungen in  $\mathfrak{M}_D$  führt auf interessante Zusammenhänge mit Levi-Civita's Parallelismus in einer Riemannschen Metrik und die isoperimetrische Ungleichung der Bolyai-Lobatscheffs-kyschen Geometrie.

M. Pinl (Köln).

**Tamura, Takayuki.** On translations of a semigroup.

Kōdai Math. Sem. Rep. 7 (1955), 67–70.

Let  $S$  be a semigroup and  $M$  the set of all single-valued mappings of  $S$  into  $S$ . If  $\mu, \nu \in M$  denote by  $\mu\nu$  the mapping obtained by following the mapping  $\nu$  by the mapping  $\mu$ . Let  $\Phi$  ( $\Psi$ ) be the subset of  $M$  consisting of the right (left) translations of  $S$ :  $\mu \in M$  is a right (left) translation of  $S$  if  $\mu(xy) = x\mu y$  ( $\mu(xy) = (\mu x)y$ ) for all  $x, y \in S$  [cf. A. H. Clifford, Trans. Amer. Math. Soc. 68 (1950), 165–173; MR 11, 499]. Each  $s \in S$  determines an inner right (left) translation  $f_s$  ( $g_s$ ):  $f_s x = xs$  ( $g_s x = sx$ ) for all  $x \in S$ . If  $R$  ( $L$ ) is the set of all inner right (left) translations of  $S$ , then  $R$  ( $L$ ) is a left ideal of the semigroup  $\Phi$  ( $\Psi$ ).

The author attempts to characterize some special types of semigroup by means of their translation semigroups. A semigroup is said to be right singular if  $xy = y$  for all  $x, y \in S$ . It is shown that  $S$  is right singular if and only if  $\Phi = M$  and if and only if  $\Psi = L = \{i_s\}$ , where  $i_s$  is the identity mapping of  $S$ . {Three theorems giving characterizations of, respectively, semigroups with one-sided identities, semigroups with a two-sided identity and zero semigroups ( $S$  is a zero semigroup if  $xy = 0$  for all  $x, y \in S$ ) are incorrect. A counter-example to each theorem is provided by the group with zero  $S = \{z, e\}$ , where  $z^2 = z$ ,  $e^2 = e$ ,  $ez = z = ze$ , and for which  $S \cong \Phi \cong R \cong \Psi \cong L$ .}

G. B. Preston (New Orleans, La.).

See also: Aleksandrow, p. 279; Gluškov, p. 280; Iséki, p. 282; Anderson, p. 325; Whitehead, p. 327; van der Woude, p. 329; Kobayashi, p. 332; Rayski, p. 359; Shibata, p. 362.

### Lie Groups, Lie Algebras

**Harish-Chandra.** The characters of semisimple Lie groups. Trans. Amer. Math. Soc. 83 (1956), 98–163.

Let  $u$  be a quasi-simple (in the sense of the author) irreducible representation of a connected semisimple Lie group  $G$ . Denote by  $C_c^\infty(G)$  the space of indefinitely differentiable complex-valued functions on  $G$  which are of compact support. If  $f \in C_c^\infty(G)$ , then the operator  $u(f) \rightarrow \int_G f(x)u(x)dx$  is of the trace class and the mapping  $T_u: f \rightarrow \text{trace of } u(f)$  is a distribution (in the sense of L. Schwartz) on the manifold  $G$ , as has been shown by the author previously. In this paper results on these characters are obtained. The central idea of the method employed is as follows. Let  $g$  be the complexification of the Lie algebra of  $G$  and  $B$  the associative enveloping algebra of  $g$ . Every element  $b$  of  $B$  can be regarded as a differential operator on  $G$  and hence acts on the distributions on  $G$ , so that the distribution  $bT_u$  is well defined. If  $z$  is in the center  $Z$  of  $B$ , then  $T_u$  is an eigendistribution of  $z$ . A certain open submanifold  $V$  of  $G$  is introduced; on it  $T^*$  defines a distribution  $t_u$ , say. The differential equation  $zT_u = \lambda T_u$  goes over into a differential equation for  $t_u$  on  $V$ ; among the latter some are always elliptic which implies that  $t_u$  is an analytic function  $V$ . The precise form of this function is studied. These results are then applied to the theory of characters. One particular case is studied

in greater detail and a striking connection between the characters of the finite- and the infinite-dimensional representations is found. A summary of the results of this paper has been published in Bull. Amer. Math. Soc. 61 (1955), 389-396 [MR 17, 173]. *F. I. Mautner.*

**Kanno, Tsuneo.** On the representations of Lie algebras.

Tôhoku Math. J. (2) 8 (1956), 46-53.

Let  $\mathfrak{P}_n = K[X_1, \dots, X_n]$  be a polynomial ring with coefficients in a field  $K$ . With each maximal ideal  $\mathfrak{M}$  in  $\mathfrak{P}_n$  the author associates an  $\mathfrak{M}$ -chain of ideals

$$\mathfrak{M}_1 \subset \dots \subset \mathfrak{M}_n = \mathfrak{M},$$

where  $\mathfrak{M}_i = \mathfrak{M} \cap \mathfrak{P}_i$  ( $1 \leq i \leq n$ ). Each  $\mathfrak{M}_i$  is a maximal ideal in  $\mathfrak{P}_i$ , and consequently  $K_{i+1} = \mathfrak{P}_i / \mathfrak{M}_i$  is a field for each  $i$ , and  $K = K_1 \subset \dots \subset K_{n+1}$ . With each  $\mathfrak{M}$ -chain is associated an  $\mathfrak{M}$ -set of irreducible polynomials  $(f_1, \dots, f_n)$ , where  $f_i \in K_i[X]$ , and  $K_i \cong K_{i-1}[X] / (f_{i-1}(X))$  for each  $i$ . There is a one to one correspondence between maximal ideals in  $\mathfrak{P}_n$  and  $\mathfrak{M}$ -sets of polynomials. Now consider a solvable Lie algebra  $\mathfrak{L}$  over an arbitrary field  $K$  of characteristic zero, and let  $\mathfrak{U}$  be the universal enveloping algebra of  $\mathfrak{L}$ . Let  $x_1, \dots, x_s, \dots, x_n$  be a basis for  $\mathfrak{L}$  such that  $x_{s+1}, \dots, x_n$  span  $[\mathfrak{L}, \mathfrak{L}]$ . We may regard  $\mathfrak{L}$  as imbedded in  $\mathfrak{U}$ , and consider the ideal  $\mathfrak{C}$  in  $\mathfrak{U}$  generated by  $[\mathfrak{L}, \mathfrak{L}]$ . Then

$$\mathfrak{U} / \mathfrak{C} \cong \mathfrak{P}_s = K[X_1, \dots, X_s],$$

and there is a one to one correspondence between equivalence classes of finite dimensional irreducible representations of  $\mathfrak{L}$  and maximal ideals in  $\mathfrak{P}_s$ . Let  $(f_1, \dots, f_n)$  be the  $\mathfrak{M}$ -set of polynomials corresponding to the maximal ideal which in turn belongs to the irreducible representation  $U: x \rightarrow U(x)$ . The correspondence is defined in such a way that  $f_i(X)$  is an irreducible factor in  $K_i[X]$  of the minimum polynomial of  $U(x_i)$  ( $i=1, \dots, s$ ). The author's second main result concerns the irreducible representations of a nilpotent Lie algebra  $\mathfrak{L}$  over an arbitrary field  $K$  of characteristic  $p > 0$ . Let  $x_1, \dots, x_n$  be a regular basis of  $\mathfrak{L}$ , such that  $[x_i, x_j] \in \sum_{k=1}^{i-1} K x_k$  whenever  $i < j$ . For each  $i$  there exists a positive integer  $s_i = p^{r_i}$  such that  $y_i = x_i^{s_i}$  belongs to the center of the universal enveloping algebra  $\mathfrak{U}$  of  $\mathfrak{L}$ . Let  $\mathfrak{B} = K[y_1, \dots, y_n]$ . Then  $\mathfrak{B}$  is isomorphic to the polynomial ring  $K[X_1, \dots, X_n]$ , and there is a one to one correspondence between the equivalence classes of irreducible representations of  $\mathfrak{L}$  and the maximal ideals in  $\mathfrak{B}$ . *C. W. Curtis (Madison, Wis.).*

See also: Hochschild, p. 278; Gluškov, p. 317; Shoenfield, p. 317; Kobayashi, p. 332.

### Topological Vector Spaces

**Helgason, Sigurdur.** A characterization of the intersection of  $L^1$ -spaces. Math. Scand. 4 (1956), 5-8.

Let an ideal  $A$  in the algebra  $C(S)$  of real-valued continuous functions vanishing at infinity on a locally compact space  $S$  satisfy the following conditions: (i) the functions in  $A$  have no common zero; (ii)  $A$  is a locally convex topological algebra and has a fundamental system of full neighborhoods of 0 ( $V$  is full if  $y \in V$ ,  $x \in A$ , and  $|x| \leq |y|$  imply  $x \in V$ ); (iii) if  $M$  is a bounded subset of  $A$  and  $g$  is in the uniform closure of  $M$ , then  $g \in A$ . Theorem: There exists a family of positive Radon measures  $\mu_k$  on  $S$  such that  $A = \bigcap_k L^1(\mu_k) \cap C(S)$ . The measures may be obtained from the continuous linear functionals on  $A$  ( $A$  contains all functions with compact support). *M. Jerison (Lafayette, Ind.).*

**Aronszajn, N.; and Smith, K. T.** Functional spaces and functional completion. Ann. Inst. Fourier, Grenoble 6 (1955-1956), 125-185.

There is assumed a class  $\mathfrak{F}$  of real or complex valued functions  $f$  on subsets of a basic set  $\mathfrak{E}$ . The sets  $A$  on which the functions  $f$  are not defined form an exceptional set  $\mathfrak{A}$ , which is assumed to be hereditary (if  $A$  is in  $\mathfrak{A}$ , and  $B \subseteq A$  then  $B$  is in  $\mathfrak{A}$ ) and  $\sigma$ -additive.  $\mathfrak{F}$  is assumed to be linear (pointwise additive except  $\mathfrak{W}$ ). It is normed with  $\|f\| = 0$  if and only if  $f = 0$  exc.  $\mathfrak{A}$ . Such a class is called a linear normed functional class relative to  $\mathfrak{A}$ . A functional space relative to  $\mathfrak{A}$ , satisfies the additional condition if  $\|f_n - f\| \rightarrow 0$ , then there exists a subsequence  $f_{n_k}$  such that  $f_{n_k}(x) \rightarrow f(x)$  exc.  $\mathfrak{A}$ . The paper is primarily concerned with the problem of functional completion i.e. the existence of a complete functional space  $\mathfrak{F}'$  relative to  $\mathfrak{A}$  such that  $\mathfrak{F}$  relative to  $\mathfrak{A}$  is embedded in  $\mathfrak{F}'$  and a dense subset of  $\mathfrak{F}'$ . In particular characterizations of minimal and maximal sets  $\mathfrak{W}'$  are desired. There are introduced the class  $\mathfrak{Q}$  of sets  $B$  of  $\mathfrak{E}$  for which there exists a function  $f$  of  $\mathfrak{F}$  such that  $|f(x)| \geq 1$  on  $B$  exc.  $\mathfrak{A}$ ,  $\delta(B) = \inf \{\|f\| \mid |f(x)| \geq 1 \text{ on } B \text{ exc. } \mathfrak{A}\}$  and  $\mathfrak{Q}_0$  the subset of  $\mathfrak{Q}$  on which  $\delta(B) = 0$ . Also the class  $\tilde{\mathfrak{Q}}$  of sets  $B$  of  $\mathfrak{E}$  for which there exist Cauchy sequences  $f_n$  of  $\mathfrak{F}$  such that  $\liminf \|f_n\| \geq 1$  on  $B$  exc.  $\mathfrak{A}$ ,  $\tilde{\delta}(B) = \inf \{\lim \|f_n\| \mid \text{for such sequences}\}$  and  $\tilde{\mathfrak{Q}}_0$  the subset of  $\tilde{\mathfrak{Q}}$  for which  $\tilde{\delta}(B) = 0$ . Then in order that  $\mathfrak{F}$  have a functional completion relative to  $\mathfrak{A}$  it is necessary and sufficient that (1) for Cauchy sequences of  $\mathfrak{F}$  converging pointwise exc.  $\mathfrak{A}$   $f_n(x) \rightarrow 0$  exc.  $\mathfrak{A}$  and  $\|f_n\| \rightarrow 0$  are equivalent, (2) every Cauchy sequence converging exc.  $\mathfrak{A}$  contains a subsequence convergent exc.  $\mathfrak{A}$ , (3) each sequence of sets  $B_n$  such that  $\tilde{\delta}(B_n) \rightarrow 0$  contains a subsequence whose limit superior belongs to  $\mathfrak{A}$ . In particular the  $\sigma$ -extension of  $\mathfrak{Q}_0$  is a lower bound for  $\mathfrak{W}'$  relative to which  $\mathfrak{F}$  has a functional completion. There is also introduced the notion of capacity. If  $\phi(t)$  is non-negative, non-decreasing on  $t \geq 0$ , continuous at  $t=0$ , then if  $B$  is in  $\mathfrak{Q}_0$  the capacity  $c_\phi = \inf \sum_{n=1}^{\infty} \phi(\delta(B_n))$  for all sequences  $B_n$  of such that  $B \subseteq \bigcup_{n=1}^{\infty} B_n$ . These capacities are of the nature of finite valued outer measures on  $\mathfrak{Q}_0$ . The principal result is that if  $\mathfrak{F}$  has a perfect functional completion (one for which every function  $g'$  equal to  $f'$  exc.  $\mathfrak{A}$  of  $\mathfrak{F}'$  also belongs to  $\mathfrak{F}'$ ), then  $\mathfrak{W}'$  is contained in  $\mathfrak{A}_0$ , where  $\mathfrak{A}_0$  is the class of all sets which are of capacity zero for all  $\phi$ . Examples applying the general theory are: (1) Analytic functions.  $\mathfrak{E}$  = unit disc in complex plane,  $\mathfrak{F}$  the class of complex valued functions continuous in  $\mathfrak{E}$  analytic in its interior and  $\|f\| = [\int_{\mathfrak{E}} |f(x)|^2 dx]^{1/2}$ .  $\mathfrak{F}$  does not act as a functional space relative to  $\mathfrak{E}$  but it does with respect to the interior of  $\mathfrak{E}$ , the exceptional class  $\mathfrak{A}$  being the class of all subsets of the boundary. (2)  $L^p$  spaces. Special treatment is given to the completion in case  $\mathfrak{E}$  is a topological space and  $\mathfrak{F}$  is the class of continuous functions which vanish outside a compact set. (3) Harmonic functions.  $\mathfrak{E}$  is the closed sphere with center 0 radius  $R$  in  $n$ -dimensional Euclidean space,  $\mathfrak{F}$  is the class of complex valued function continuous on  $\mathfrak{E}$  and harmonic interior to  $\mathfrak{E}$  with  $\|f\| = [\int_{\partial \mathfrak{E}} |f(\theta)|^2 d\theta]^{1/2}$ ,  $\partial \mathfrak{E}$  = boundary of  $\mathfrak{E}$ . The well-known theorem of Fatou on the boundary values of a harmonic function is proved via the capacities. (4) Riesz potentials.  $\mathfrak{E}$  is Euclidean  $n \geq 2$  space,  $\mathfrak{F}_\alpha$  is the class of all differences of positive Borel measures  $\mu$  for which  $\|\mu\|^2 = \iint K_\alpha(x-y) d\mu(x) d\mu(y) < \infty$ , where

$$K_\alpha(x) = |x|^{n-\alpha} / H_n(\alpha); \quad H_n(\alpha) = \pi^{n/2} \Gamma(\alpha/2) / \Gamma((n-\alpha)/2),$$

$0 < \alpha < n$ . The exceptional sets  $\mathfrak{A}_\alpha$  are the sets  $A$  for which

there exists a positive measure  $\mu$ , such that the integral  $\int K_\alpha(x-y)d\mu(y)$  is infinite for all  $x$  of  $A$ . It is shown that  $\mathfrak{H}_\alpha$  is a functional space relative to  $\mathfrak{H}_\alpha$  and that it has a functional completion relative to  $\mathfrak{H}_\alpha$ . This last example is related to papers of H. Cartan [C. R. Acad. Sci. Paris 214 (1942), 944-946, 994-997; Bull. Soc. Math. France 69 (1941), 71-96; 73 (1945), 74-106; MR 5, 146; 7, 447] and Deny [Acta Math. 82 (1950), 107-183; MR 12, 98].

T. H. Hildebrandt (Ann Arbor, Mich.).

Iohvidov, I. S.; and Krein, M. G. Spectral theory of operators in space with indefinite metric. I. Trudy Moskov. Mat. Obšč. 5 (1956), 367-432. (Russian)

A Hilbert space  $\Pi_n$  with an indefinite inner product is discussed. A basic axiom is the decomposability of  $\Pi_n$  into a direct sum of two subspaces  $\Pi_+$  ( $n$ -dimensional) and  $\Pi_-$  (the orthogonal complement of  $\Pi_+$ ) such that the inner product is positive definite on  $\Pi_+$  and  $\Pi_-$ , provided with the norm  $|x| = \sqrt{-(x, x)}$ , is a Banach space. Furthermore, it is assumed that no subspace on which  $(x, x)$  is positive definite can have dimension above  $n$ . It is then shown that if  $P$  is an arbitrary subspace of dimension  $n$ , on which  $(x, x)$  is positive definite, then its orthogonal complement  $N$ , normed by  $|\cdot|$  is a Banach space. If  $\|x\| = \sqrt{[x, x]} = \sqrt{((x^P, x^P) - (x^N, x^N))}$ , where  $x^{PN}$  are the components of  $x$  relative to  $P$  and  $N$ , then  $\|\cdot\|$  and  $|\cdot|$  are equivalent on  $N$ .

A linear set  $L$  is called degenerate in case  $L \cap L^\perp \neq 0$  and  $\neq 0$ . If  $L$  is nondegenerate so is  $L^\perp$  and  $L^{\perp\perp} = L$ . If  $L$  is nondegenerate and  $\bar{L}$  is the closure of  $L$  relative to  $\|\cdot\|$ , then  $\bar{L}$  is nondegenerate. If  $\Pi_n$  is separable, it possesses a semiorthonormal basis  $e_i$ ,  $(e_i, e_j) = 0$ ,  $i \neq j$ ,  $(e_i, e_i) = -1$ ,  $1 \leq i \leq n$ ,  $(e_i, e_i) = 1$ ,  $i > n$ .

A Riesz type representative for linear functionals on  $\Pi_n$  is obtained:  $f \in \Pi_n^* \Rightarrow f(x) = (x, y)$ ,  $y \in \Pi_n$ . Hence the usual properties and types of operators are definable, and the expected properties of isometric, self-adjoint, etc., operators are derived. (A variation: the spectrum of a unitary operator  $U$  is symmetric with respect to the unit circle:  $\lambda \in \text{spectrum } U \Rightarrow \bar{\lambda}^{-1} \in \text{spectrum } U$ .)

Cayley transforms of closed operators are discussed and the standard results are proved. There is a discussion of defect indices.

A reduction theory seems to be limited to special operators and finite dimensional reducing subspaces: If  $(Tx, Tx) > (x, x)$  for all  $(x, x) \geq 0$ ,  $x \neq 0$ , there is an  $n$ -dimensional nondegenerate subspace  $T$  in which all the proper values of  $T$  are at least 1 in absolute value. Consequently, if  $U$  is unitary, there are two finite dimensional subspaces  $T$  and  $T'$  which reduce  $U$  and the proper values of  $U$  in  $T$  are all less or equal to 1 in absolute value, and the proper values of  $U$  in  $T'$  have modulus not less than 1.

The finite dimensional reducing spaces of an arbitrary self-adjoint operator are then deduced via Cayley transforms. If  $(Tx, Tx) > (x, x)$ , all  $x \neq 0$ , the above can be specialized, in particular, if  $\Pi_n$  is of dimension  $N$ . Then  $\Pi_n$  can be split into an  $n$ - and an  $(N-n)$ -dimensional subspace, each of which reduces  $T$ .

Finally, a "general form" for unitary and semiunitary operators is derived.

B. Gelbaum.

Pták, Vlastimil. On a theorem of Mazur and Orlicz. Studia Math. 15 (1956), 365-366.

The following theorem subsumes many of its predecessors and is almost its own proof. Let  $X$  be a linear space,  $T$  a set,  $x(t)$  a mapping of  $T$  into  $X$ ,  $\beta(t)$  a real-

valued function on  $T$ . Let  $\omega(x+y) \leq \omega(x) + \omega(y)$ ,  $\omega(\lambda x) = \lambda\omega(x)$ ,  $\lambda \geq 0$ ,  $x, y \in X$ . Then a necessary and sufficient condition that there exists an additive homogeneous function  $f(x)$  on  $X$  such that  $f(x) \leq \omega(x)$ ,  $\beta(t) \leq f(x(t))$  is  $\sum \lambda_i \beta(t_i) \leq \omega(\sum \lambda_i x(t_i))$  for all finite sets  $\lambda_1, \lambda_2, \dots, \lambda_n$  and  $t_1, t_2, \dots, t_n$ . The function

$$\tilde{\omega}(x) = \inf\{\omega(x + \sum \lambda_i x(t_i) - \sum \lambda_i \beta(t_i)) \mid \text{all finite sets } \{\lambda_i\}, \{t_i\}\}$$

is subadditive and non-negative homogeneous. Thus there is an  $f(x) \leq \tilde{\omega}(x)$  and clearly  $\tilde{\omega}(x) \leq \omega(x)$ . Obviously,  $\beta(t) \leq f(x(t))$ . B. Gelbaum (Minneapolis, Minn.).

Singer, Ivan. Sur l'extension des fonctionnelles linéaires.

Rev. Math. Pures Appl. 1 (1956), no. 2, 99-106.

If  $M$  is a subspace of a Banach space  $E$ , and  $\varphi \in M^*$ , then

$$\mathcal{F}_\varphi = \{f \mid f \in E^*, \|f\| = \|\varphi\|, f = \varphi \text{ on } M\}.$$

If  $\Phi CM^*$ ,  $\mathcal{F}_\varphi = \bigcup_{\Phi} \mathcal{F}_\varphi$ . Results:  $\mathcal{F}_\varphi$  is convex, weakly closed, regularly convex, respectively, and extreme on the unit sphere of  $E^*$  if and only if  $\Phi$  enjoys these properties respectively in  $M^*$ . [Note: an extreme subset  $B$ , of a convex closed set  $A$ , is any subset which fails to contain the interior of any segment which lies in  $A$  but is not contained in  $B$ .] Thus  $\mathcal{F}_\varphi$  is extreme if and only if  $\varphi$  is an extreme point. Furthermore, if  $\varphi$  is extreme, there is an extension  $f$  of  $\varphi$ ,  $\|f\| = \|\varphi\|$  which is also extreme. Hence, too, if  $e(S)$  is the cardinality of the extreme points of the sphere  $S$ , then  $e(S_{M^*}) \leq e(S_{E^*})$ . There can be an  $f$  which extends a  $\varphi$ ,  $\|f\| = \|\varphi\|$ , such that  $f$  is extreme and  $\varphi$  is not.

These results are announced to be ancillary to the author's research on optimal polynomial approximation in Banach spaces. B. Gelbaum (Minneapolis, Minn.).

Monna, A. F. Sur les espaces normés non-archimédiens.

I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 59= Indag. Math. 18 (1956), 475-483, 484-489.

Let  $L$  denote a normed linear space over a field  $K$  with a non-archimedean valuation, and  $\|x+y\| \leq \max(\|x\|, \|y\|)$ ,  $x, y \in L$ . It is assumed, moreover, that the valuation in  $K$  is not improper, and that  $K$  is complete with respect to the valuation. The papers are concerned with the theory of projectors and orthogonality in  $L$ . Use is made of the notion of spherical completeness; that is, that each ordered (with respect to inclusion) family of "balls" of  $L$  has a non-empty product, where a ball  $B(x_0, \rho) = \{x \in L \mid \|x - x_0\| \leq \rho\}$ ,  $x_0 \in L$ ,  $\rho = \|x\|$  for some  $x \in L$ .

L. M. Blumenthal (Columbia, Mo.).

Bonsall, F. F. Extreme maximal ideals of a partially ordered vector space. Proc. Amer. Math. Soc. 7 (1956), 831-837.

For a subspace  $V_0$  of a partially ordered vector space  $V$  with order-unit  $e$ , let  $H(V_0)$  denote the set of all  $x \in V$  such that for each real  $\epsilon > 0$  there exists  $w_\epsilon \in V_0$  having  $-(\epsilon e + w_\epsilon) \leq x \leq \epsilon e + w_\epsilon$ . An ideal  $J$  of  $V$  is said to be perfect provided  $JCH(J)$ . It is proved that a maximal ideal is perfect if and only if it is extreme; a corollary is the fact that if  $V$  is lattice-ordered, then the extreme maximal ideals coincide with the lattice maximal ideals. The class of all perfect ideals shares important properties of the class of all ideals. In particular, if  $(0)$  is the only proper perfect ideal, then  $V$  is one-dimensional; and every proper perfect ideal is contained in a perfect maximal ideal. From this last result is deduced a slightly generalized form of the Krein-Milman theorem, proved earlier by the author [Proc. London Math. Soc. (3) 4 (1954), 402-418; MR 16, 936]. V. L. Klee, Jr.

Nef, Walter. **Monotone Linearformen auf teilgeordneten Vektorräumen.** Monatsh. Math. 60 (1956), 190-197.

A linear form on a partially ordered linear space is called monotonic if it is isotonic, i.e. non-negative for all positive elements. The principal results are: A. A linear form  $F$  defined on a subspace  $L$  can be extended to a monotonic linear form on the entire space if and only if there is a bounded convex set  $K$ , which is such that the sets  $\lambda K$  for positive  $\lambda$  span the space, and such that, for every  $y$  in the space, the set of values of  $F$  on the points of  $L$  which are greater than some point in  $y+K$  are bounded below; B. A monotonic linear form, not identically zero, exists on the space if and only if there is a one-dimensional subspace which is not majorized; C. For any  $e$  in the space, such a form exists and satisfies  $F(e)=1$  if and only if the space generated by  $e$  is not majorized.

J. L. B. Cooper (Cardiff).

Baluev, A. N. **On the method of Čaplygin.** Vestnik Leningrad. Univ. 11 (1956), no. 13, 27-42. (Russian)

This paper is concerned with the existence of solutions of the equation  $P(x)=0$  where  $P$  is a (non-linear) function from the partially ordered linear space  $X$  to the partially ordered linear space  $U$  and where  $X$  and  $U$  satisfy axioms I, II, III and IV formulated by Kantorovich, Vulich and Pinsker in their book "Functional analysis in partially ordered spaces" [Gostehizdat, Moscow, 1950; MR 12, 340]. The basic lemma (Theorem 1) is an abstract formulation of a principle used by Čaplygin in studying the existence of solutions of differential equations. It may be stated as follows. Let there exist elements  $x_0 \leq y_0$  in  $X$  and additive homogeneous operators  $T_1$  and  $T_2$  from  $X$  to  $U$  such that (a)  $P(x_0) \leq 0 \leq P(y_0)$ , (b)  $T_1^{-1}$  and  $T_2^{-1}$  exist and are non-negative, and (c)  $P(y_0) + T_2(x - y_0) \leq P(x) \leq P(x_0) + T_1(x - x_0)$  for all  $x$  in the interval  $[x_0, y_0]$ . Then if we set  $x_1 = x_0 - T_1^{-1}(P(x_0))$  and  $y_1 = y_0 - T_2^{-1}(P(y_0))$  we have  $x_0 \leq x_1 \leq y_1 \leq y_0$ ,  $P(x_1) \leq 0 \leq P(y_1)$  and every solution of  $P(x)=0$  in the interval  $[x_0, y_0]$  is also in the interval  $[x_1, y_1]$ . A typical existence theorem proved using the basic lemma recursively is the following. Suppose that  $X$  and  $U$  satisfy in addition axiom V of the above quoted book for countable sets and that  $P$  is uniformly differentiable in the sense defined in a recent paper of Kantorovich [Dokl. Akad. Nauk SSSR (N.S.) 76 (1951), 17-20; MR 12, 835] in every interval  $[x, y]$ . Let  $x_0$  and  $y_0$  be points of  $X$  such that  $x_0 \leq y_0$  and  $P(x_0) \leq 0 \leq P(y_0)$  and suppose that  $P'(x) \leq T(x)$  on  $[x_0, y_0]$  where  $T$  is additive and homogeneous, has a positive inverse and at least one of  $T$  and  $T^{-1}$  is continuous. Then  $P(x)=0$  has at least one solution on  $[x_0, y_0]$ . There are other similar existence theorems and a discussion of how some of the hypotheses may be formulated in case the spaces are normed.

G. W. Mackey (Cambridge, Mass.).

Nikol'skii, S. M. **Compactness of classes  $H_p^{(r_1, \dots, r_n)}$  of functions of several variables.** Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 611-622. (Russian)

The classes of the title have been used by the author in several connections [see, e.g., Trudy Mat. Inst. Steklov., v. 38 (1951), pp. 244-278; MR 14, 32]. Roughly speaking, if  $r_i = r_i + \alpha_i$ ,  $0 < \alpha_i \leq 1$ , they consist of functions whose  $r_i$ th derivatives with respect to  $x_i$  satisfy  $L^p$  Lipschitz conditions of respective orders  $\alpha_i$  (if  $\alpha_i < 1$ ) or the corresponding symmetric conditions (as for "smooth" functions) if  $\alpha_i = 1$ ; the integration is over a domain  $G$  in  $n$ -

space. If  $M$  is the constant appearing in the Lipschitz conditions, the norm of  $f$  is taken to be its  $L^p$  norm plus  $M$ , and the space is a Banach space. The author also considers a similar class  $W_p^{(\rho)} H^{(\alpha)}$ , where all the partial derivatives (mixed ones included) of order  $\rho$  satisfy an  $L^p$  Lipschitz condition of order  $\alpha$ . The compactness theorem states that (again roughly) if we have a set of elements of one of these spaces with norms bounded by  $K$ , there is a subsequence converging over every interior subregion, in the norm of the space obtained by replacing  $r_i$  by smaller numbers  $r'_i$ , to an element of the original space with norm not exceeding  $K$ . For  $W_p^{(\rho)} H^{(\alpha)}$  the convergence is in the norm of  $W_p^{(\rho)} H^{(\beta)}$  with  $\beta < \alpha$ . The proof depends on the following generalization of a lemma of Mazur's [Banach, Théorie des opérations linéaires, Warsaw, 1932, p. 237]. Let  $E$  and  $E_*$  be normed linear spaces consisting of the same elements  $x$  with  $\|x\|_* \leq c\|x\|$ . Let  $F$  be a bounded set in  $E$  and let  $A_n(x)$  be operators mapping  $E$  into  $E_*$ , such that  $A_n(x) = x - U_n(x)$ ;  $U_n(x)$  is completely continuous (not necessarily linear);  $\sup_{x \in F} \|A_n(x)\| \rightarrow 0$ . Then  $F$  is sequentially compact in  $E_*$ . R. P. Boas, Jr.

Zygmund, A. **On a theorem of Marcinkiewicz concerning interpolation of operations.** J. Math. Pures Appl. (9) 35 (1956), 223-248.

To M. Riesz and G. O. Thorin [Thorin, Kungl. Fysiogr. Sällsk. i Lund Förh. 8 (1938), 166-170] is due the following theorem. Let  $T$  be a linear bounded operator from  $L^a(\mu)$  to  $L^b(\nu)$ ,  $\|f\|_{a,b} = [\int_R |f|^a d\mu]^{1/a}$  and  $M_{a,b}$  its bound. Then the assertion is that if  $T$  is a bounded operator for  $a_1, b_1$  and  $a_2, b_2$ , then it is a bounded operator for  $a_3, b_3$ , where

$$\frac{1}{a_3} = \frac{t}{a_1} + \frac{1-t}{a_2}, \quad \frac{1}{b_3} = \frac{t}{b_1} + \frac{1-t}{b_2},$$

$$M_{a_3, b_3} \leq M_{a_1, b_1}^t M_{a_2, b_2}^{1-t}.$$

Below is given the definition of "weak boundedness" (reviewer's term) which is such that if  $T$  is weakly bounded for  $a_1, b_1$  and for  $a_2, b_2$ , then  $T$  is bounded for  $a_3, b_3$  given as above. In this case  $T$  may not even be linear, but satisfy only  $\|T(f_1 + f_2)\| \leq \kappa(\|Tf_1\| + \|Tf_2\|)$ ,  $\kappa$  independent of  $f_i$ . Also  $M_{a_3, b_3} \leq K M_{a_1, b_1}^t M_{a_2, b_2}^{1-t}$ , where  $K$  depends only on  $a_i, b_i, \kappa$  and stays bounded, if  $t$  stays away from 0 and 1. This is the theorem of Marcinkiewicz in a form given and proved in this paper. It remains to give the definition of weak boundedness.  $T$  will be called weakly bounded with respect to  $a, b$ , if there exists an  $M_{a,b}$  such that  $\nu\{t | |Tf(t)| > y\} \leq (M_{a,b} \|f\|_{a,b}/y)^b$ .

Numerous corollaries are given, especially some to spaces whose norm is of the form  $\int \varphi(|f|) dx$ , and to Fourier expansions into orthonormal series. If  $T$  is a Hilbert transform, then it is well-known, that it is bounded  $p, p$  for all  $p > 1$ . It is interesting to note that it is weakly bounded 1, 1. František Wolf.

Gates, Leslie D., Jr. **Linear differential equations in distributions.** Proc. Amer. Math. Soc. 7 (1956), 933-939.

$L[T]$  denotes  $p_0 T^{(n)} + p_1 T^{(n-1)} + \dots + p_n T$ , where  $T$  is an unknown distribution (in the sense of L. Schwartz), and the  $p_i$  are point functions with derivatives of all orders. Under stated conditions the author finds a solution for the equation  $L[T] = S$ , where  $S$  is a given distribution. The method gives, in particular, the complete solution for the equation  $xT' + T = \delta$ . The author uses the relation

$L[T] \cdot \varphi = T \cdot \bar{L}[\varphi]$  for testing function  $\varphi$ , where

$$\bar{L}[\varphi] = \sum (-1)^{n-i} (\varphi, \varphi)^{(n-i)}.$$

I. Halperin (Kingston, Ont.).

Sard, Arthur. Approximation and projection. J. Math. Phys. 35 (1956), 127-144.

This paper, similar to the author's earlier one [Trans. Amer. Math. Soc. 73 (1952), 426-446; MR 14, 658] but with a somewhat different formulation, considers the problem of choosing a linear operator  $T$  from a given linear manifold  $\mathfrak{F}$  of bounded linear operators on a Hilbert space  $X$  such that  $\|Tg - h\|$  attains a minimum over  $T$  for fixed  $g$  and  $h \in X$ , such  $T$  being called efficient. Trivially  $T$  is efficient with  $Tg = u$  if and only if  $u \in \mathfrak{F}g$  for  $u$  the orthogonal projection of  $h$  onto the closure of  $\mathfrak{F}g$ . By taking  $X$  to be the direct product  $Z \# L_2(\Omega, \rho)$ , where  $(\Omega, \rho)$  is a probability measure space and  $Z$  another Hilbert space, the author also discusses "strongly efficient" operators and interprets the above problem as finding an optimal approximation to a true signal from a contaminated one, or alternatively as prediction theory. It appears to the reviewer that the significance of the author's results would be easier to judge if he had worked out the details of a few specific examples for these applications.

F. H. Brownell (Seattle, Wash.).

Sobolev, S. L. Remarks on the numerical solution of integral equations. Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 413-436. (Russian)

The paper begins by describing the notion of the closure of a numerical algorithm; thus the relaxation algorithm for solving a boundary value problem for a partial differential equation has as its closure a set of equations describing the 'diffusion' of the boundary values over the whole domain. The closure of an algorithm is said to be regular if the linear operators describing it are bounded in the appropriate function spaces.

The author applies these ideas to the solution of linear integral equations of the second kind by algorithms in which (i) the equation is approximated by a finite set of linear equations in a finite number of variables, and (ii) the latter system is solved by successive elimination of the unknowns.

To analyse part (i) of the process, the author defines a regular approximation to a completely continuous operator  $A$  as being a uniformly completely continuous sequence  $(A_n)$  of operators converging strongly to  $A$ . He proves that if  $(A_n)$  is such a sequence, and  $(I - A)^{-1} = I + \Gamma$  exists, then  $(I - A_n)^{-1} = I + \Gamma_n$  exists for all sufficiently large  $n$ , and  $(\Gamma_n)$  is a regular approximation to  $\Gamma$ .

Now let  $A$  be the operator defined by a continuous kernel  $K(x, y)$  in the space  $C[0, 1]$ , and let

$$A_N \varphi = \sum_{n=1}^N K_n(x) \varphi(t_n) \quad (N \geq 1),$$

where  $t_n = (n - \frac{1}{2})/N$ ,  $K_n(x) = hK(x, t_n)$ ,  $h = (N + 1)^{-1}$ ; then  $(A_N)$  is a regular approximation to  $A$ . Other regular approximations to integral operators are also described.

The closure of the above algorithm is described by equations of the form

$$\varphi(x) - \int_0^1 \Gamma(x, y, z) \varphi(y) dy = f(x) + \int_0^z \Gamma(x, y, z) f(y) dy,$$

where  $\Gamma(x, y, 0) = K(x, y)$ ,  $\Gamma(x, y, 1) = \Gamma(x, y)$  and  $z$  runs from 0 to 1. The author shows that the closure is regular

if and only if none of the equations

$$\varphi(x) - \int_0^z K(x, y) \varphi(y) dy = \varphi(x),$$

where  $0 < z \leq 1$ , is singular; when this holds, and only then, the algorithm described will give results of arbitrarily high accuracy for sufficiently large  $N$ . A simple example is given of an equation for which the closure of this algorithm is not regular. The paper concludes with some formal properties of the family of kernels  $\Gamma(x, y, z)$ .

F. Smithies (Cambridge, England).

See also: Schwartz, p. 287; Krumbach, p. 288; Choquet, p. 288; Morgenstern, p. 289; Gagaev, p. 301; Haas, p. 305; Harish-Chandra, p. 318; Pukánszky, p. 323; Berezanskii, p. 323; Klee, p. 330.

### Banach Spaces, Banach Algebras

Zuhovickii, S. I. On a minimum problem in the space of continuous functions. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 383-384. (Russian)

Let  $E$  be a linear normed space and  $G$  a subspace of  $E$ . In a recent paper [Math. Z. 63 (1955), 97-108; MR 17, 273] Rogosinski studies properties of minimal extensions to  $E$  of linear continuous functionals defined on  $G$ ; he confines himself to the case when  $E$  is either  $l^p$  ( $p \geq 1$ ) or  $c$ . In the present note the author considers a parallel problem when  $E$  is the space  $C(a, b)$  of functions  $x(t)$  continuous on  $[a, b]$ , and  $\|x\| = \max_t |x(t)|$ . He observes, for example, that if a linear functional  $\varphi(x)$  defined in  $GCC(a, b)$  has a maximal element  $X(t) \in G$  (i.e.  $X| = 1$ ,  $\varphi(X) = \|\varphi\|$ ), then the kernels  $g(t)$  of all minimal extensions  $f(x) = \int_a^b x(t) dg(t)$  of  $\varphi$  have a similar structure in the sense that all  $g(t)$  are constant on all subintervals of  $[a, b]$ , where  $|X(t)| < 1$  and are non-decreasing at each point of the subset of  $(a, b)$  where  $X(t) = +1$ . Remarks about necessary and sufficient conditions for the existence of a maximal element for the functional  $f(x) = \int_a^b x(t) dg(t)$  in  $C(a, b)$ . A. Zygmund.

Donoghue, William F., Jr. The Banach algebra  $l^1$  with an application to linear transformations. Duke Math. J. 23 (1956), 533-537.

Soit  $l^1$  l'algèbre de Banach formée des fonctions  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  analytiques pour  $|z| < 1$  et dont la série de Taylor est absolument convergente sur  $|z| = 1$ , avec la norme  $\|f\| = \sum_{n=0}^{\infty} |a_n|$ . L'auteur démontre que si  $I$  est un idéal fermé dans  $l^1$ , pour que  $l^1/I$  soit réflexif, il faut et il suffit que  $I$  soit de codimension finie. Il en déduit la conséquence suivante: soient  $E$  un espace de Banach réflexif de dimension infinie,  $T$  un endomorphisme continu de  $E$  tel que  $\|T^n\| \leq h$  (borne indépendante de  $n$ ). Alors, pour tout  $x \in E$ , l'enveloppe cerclée convexe fermée de l'ensemble des  $T^n x$  ne contient aucun point intérieur. Un contre-exemple montre que l'hypothèse que les  $\|T^n\|$  sont bornés est essentielle. J. Dieudonné (Evanston, Ill.).

Widom, Harold. Approximately finite algebras. Trans. Amer. Math. Soc. 83 (1956), 170-178.

The notion of approximate finiteness, originally defined by Murray and von Neumann for factors of types  $II_1$  on separable Hilbert space, is here generalized to the non-separable case, and in fact to type  $II$   $AW^*$ -algebras with central trace. Let  $M$  be such an algebra,  $Z$  its center;  $M$  is then called approximately finite (A) if given elements

$A_1, \dots, A_n \in M$  and  $\varepsilon > 0$  there exists an  $AW^*$  subalgebra  $N$  of  $M$ , with center  $Z$ , and elements  $B_1, \dots, B_n$  of  $N$  such that  $\text{tr}((A_i - B_i)^*(A_i - B_i)) < \varepsilon$  ( $i=1, \dots, n$ ). Factors of type  $II_1$  obtained as quotient algebras of finite type I  $AW^*$ -algebras by maximal ideals are shown not to be approximately finite (A).  $M$  is called approximately finite (B) if it is generated by mutually commuting  $AW^*$ -subalgebras of type I, all having  $Z$  again as center. Then approximate finiteness (B) is shown to imply approximate finiteness (A), and the author remarks that in a paper yet to appear he shows that the converse does not hold. A structure theorem is given for algebras  $M$  which are approximately finite (B), best illustrated when  $M$  is a factor. Let  $\chi(M)$  be the minimum cardinal number of commuting subalgebras of type I with center  $Z$  which can generate  $M$ . Then if the factors  $M_1$  and  $M_2$  are approximately finite (B),  $\chi(M_1) = \chi(M_2)$  if and only if  $M_1$  and  $M_2$  are approximately finite. Some of these results were also obtained by Y. Misonou, Tôhoku Math. J. (2) 7 (1955), 192, 205; MR 17, 990.

J. Feldman.

**Pukánszky, L.** Some examples of factors. Publ. Math. Debrecen 4 (1956), 135-156.

This paper is concerned with factors of type III and the objective is to extend certain results known for type II to these. It is shown that there exists non-isomorphic factors of type III. A second major result is that there exists maximal semi-regular abelian rings in the sense of Dixmier in some factors of type III.

A factor  $M$  is said to have the property  $L$  if there exists a sequence of unitary operations  $U_n, U_n \in M$ , such that  $\text{weak } \lim U_n = 0$  and  $\text{strong } \lim U_n^* A U_n = A$  for every  $A \in M$ . Two factors of type III are constructed, one of which has this property  $L$ , the other does not. Thus these factors are not isomorphic.

Let  $N$  be a subring of  $M$  and  $T$  denote the ring determined by those unitary operators of  $M$  for which  $U^* N U \subset N$ . Dixmier calls  $N$  "semiregular" if  $T$  is a proper subset of  $M$ . In the present paper it is shown that there exists factors of type III which have maximal abelian subrings which are semiregular.

The construction of the examples is based on the von Neumann procedure [Ann. of Math. (2) 41 (1940), 94-161; MR 1, 146]. A measure space  $X$  is given and a countable group  $\mathcal{G}$  of transformations of  $X$  into itself. Under certain circumstances a system of group numbers of  $\mathcal{G}$  with coefficients which are measurable functions on  $X$  can be set up both to represent the Hilbert space  $\mathfrak{H}$  and the ring  $M$ .

F. J. Murray (New York, N.Y.).

See also: Zeller, p. 301; Young, p. 316; Singer, p. 320; Nikol'skiĭ, p. 321; Sobolev, p. 322.

## Hilbert Space

**Berezanskii, Yu. M.** On expansion according to eigenfunctions of general self-adjoint differential operators. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 379-382. (Russian)

Let  $\sigma$  be a measure on the real line and  $N(\lambda)$  a dimension function. The direct integral  $L^2(\sigma, N)$  is by definition a Hilbert space consisting of the vector-valued functions  $F(\lambda) = \{F_k(\lambda)\}_{k=1}^{N(\lambda)}$  with the scalar product

$$(F, G) = \int F(\lambda) \cdot \overline{G(\lambda)} d\sigma(\lambda),$$

$(F(\lambda) \cdot \overline{G(\lambda)}) = \sum F_k(\lambda) \overline{G_k(\lambda)}$ . Let  $A$  be a self-adjoint operator on a separable Hilbert space  $H$ . The spectral theorem may be stated in the following form: there exists a direct integral  $H^* = L^2(\sigma, N)$  and a unitary mapping  $U$  from  $H$  to  $H^*$  which diagonalizes  $A$  in the sense that  $U A U^{-1}$  is multiplication by  $\lambda$ . Now let  $H$  be all square integrable functions on an open subset  $S$  of real  $n$ -space. The author shows that for almost all  $\lambda$ , the components of  $F(\lambda) = (Uf)(\lambda)$  are distributions considered as functions of  $f$ . Slightly modified, the proof runs as follows. Let  $b$  be the differential operator  $(-\Delta)^k + 1$  ( $2k > n$ ). It has a square integrable solution  $B(x)$  defined in the entire space. Applying Parseval's formula to the identity  $(f, g) = (Bbf, g)$ ,  $((Bf)(x) = \int B(x-y)f(y)dy)$ , we get

$$(F, G) = \int \left( \int C(\lambda, y) f(y) dy \right) \cdot \overline{G(\lambda)} d\sigma(\lambda)$$

where  $f \in C^\infty$  vanishes outside a compact subset of  $S$ ,  $G = Ug$  and  $C(\lambda, y) = U_x B(x-y)$ . Because  $g$  is arbitrary,

$$(Uf)(\lambda) = F(\lambda) = \int C(\lambda, y) b f(y) dy$$

is a distribution for almost all  $\lambda$ . When  $S$  is the entire space,

$$(1) \quad \iint |C(\lambda, y)|^2 d\sigma(\lambda) dy / (1 + |y|^{n+\varepsilon}) < \infty \text{ if } \varepsilon > 0,$$

which means that  $\int |C(\lambda, y)|^2 dy / (1 + |y|^{n+\varepsilon})$  is finite for almost all  $\lambda$ . (The author gets the exponent  $2n+1+\varepsilon$  because of another choice of  $b$ ). The results are specialized to the case when  $A$  is the restriction of a differential operator  $A_0$  (considered as a distribution) with sufficiently differentiable coefficients. When  $A_0$  is elliptic,  $b(D_y)C(\lambda, y)$  exists and is an ordinary eigenfunction of  $A_0$  and (1) can be improved. Finally, explicit formulas are given when  $S$  is the entire space,  $n=2$  and  $A_0 = \partial^2/\partial x_1^2 - \partial^2/\partial x_2^2$ .

L. Gårding (Lund).

**de Sosa Páez, Susana Z.; and Muñoz, Lina N.** Geometry of the sphere in Hilbert space. Rev. Un. Mat. Argentina 17 (1955), 279-286 (1956). (Spanish)

See also: Sasaki, p. 275; Morgenstern, p. 289; Wisdom, p. 322; de Groot, p. 325.

## TOPOLOGY

### General Topology

**Heppes, A.; und Révész, P.** Zum Borsukschen Zertheilungsproblem. Acta Math. Acad. Sci. Hungar. 7 (1956), 159-162. (Russian summary)

Borsuk's conjecture that every subset of euclidean  $n$ -space, with diameter  $D$ , is decomposable into  $n+1$  sets, each with diameter less than  $D$ , was proved by Eggleston for  $n=3$  [J. London Math. Soc. 30 (1955), 11-24; MR 16,

734]. This paper offers a simpler proof for the special case of a finite subset of three-space. L. M. Blumenthal.

**Mycielski, Jan.** Generalizations of the theorems on paradoxical decompositions of the sphere. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 199-200.

En se servant d'un résultat de W. Sierpiński [Fund. Math. 33 (1945), 235-244; MR 8, 140] sur le groupe de rotations de la sphère, l'auteur discute la décomposition

de la sphère en un nombre transfini de parties disjointes, et généralise les résultats de R. M. Robinson [ibid. 34 (1947), 246-260; MR 10, 106] et J. F. Adams [J. London Math. Soc. 29 (1954), 96-99; MR 15, 691]. Les démonstrations de ces résultats seront trouvées dans l'oeuvre analysée ci-dessous. *M. Kondô* (Paris).

**Mycielski, Jan.** On the paradox of the sphere. Fund. Math. 42 (1955), 348-355.

Dans cette note, en généralisant les résultats de Sierpiński [voir l'analyse ci-dessus], de Robinson [Fund. Math. 34 (1947), 246-260; MR 10, 106] et de Adams [J. London Math. Soc. 29 (1954), 96-99; MR 15, 691], l'auteur a démontré les théorèmes suivants. (T<sub>1</sub>) Il existe un ensemble  $E$  sur une sphère  $S$  tel que, pour chaque nombre cardinal  $n$ , où  $2 < n \leq 2^{\aleph_0}$ ,  $S$  peut être décomposée en  $n$  ensembles disjoints et congruents à  $E$  par les rotations de  $S$ . (T<sub>2</sub>) Soient  $M$  et  $N$  deux ensembles tels que  $0 < \bar{M} \leq 2^{\aleph_0}$ , et  $0 < \bar{N} \leq 2^{\aleph_0}$ , et  $\{q_\mu\}_{\mu \in M}$   $m$  rotations indépendantes d'une sphère  $S$ , où  $m = \bar{M}$ . Alors, pour un système de congruences  $\sum_{\mu \in P_\mu} A_\mu \simeq \sum_{\mu \in Q_\mu} A_\mu$  ( $\mu \in M$ ), où  $\{P_\mu\}_{\mu \in M}$  et  $\{Q_\mu\}_{\mu \in M}$  sont deux classes de sous-ensembles non vides de  $N$ , il existe un sous-ensemble  $R$  de  $S$  tel que  $\bar{R} = 2^{\aleph_0}$  et qui peut être décomposé en  $n$  ensembles disjoints  $\{A_\nu\}_{\nu \in N}$  remplissant les relations  $\varphi_\mu(\sum_{\nu \in P_\mu} A_\nu) = \sum_{\nu \in Q_\mu} A_\nu$  ( $\mu \in M$ ). (T<sub>3</sub>) Il existe une suite  $\{A_n\}$  ( $n=1, 2, \dots$ ) de sous-ensembles non vides et disjoints d'une sphère  $S$  telle que

$$\sum_{n \in N_1} A_n \simeq \sum_{\text{rot } n \in N_2} A_n$$

pour chaque ensembles non vides  $N_1$  et  $N_2$  de nombres naturels.

Le théorème (T<sub>1</sub>) est prononcé sans la démonstration dans la note analysée ci-dessus et le théorème (T<sub>2</sub>) est une amélioration du théorème 3 de la même note. *M. Kondô*.

**Kapruano, Isaac.** Points accessibles et corps de nombres complexes. C. R. Acad. Sci. Paris 243 (1956), 546-549.

The author continues his study [same C. R. 240 (1955), 2193-2196; 242 (1956), 978-981, 1833-1836, 2614-2617; MR 17, 874, 1065, 1189] of fields of complex numbers. A major theorem gives properties of accessible points by simple arcs of hereditarily indecomposable snake-like [caténal] continua in the plane. *W. R. Utz*.

**Nagata, Jun-iti.** On coverings and continuous functions. J. Inst. Polytech. Osaka City Univ. Ser. A. 7 (1956), 29-38.

This is a more detailed version of an earlier paper by the author [Proc. Japan Acad. 31 (1955), 688-693; MR 17, 650]; full proofs are given, and some new corollaries are added. Some of the principal results, which were described but not explicitly stated in the review of the earlier paper, are as follows (with  $R$  denoting a Hausdorff space, and  $\{f_\alpha\}$  a family of continuous, real-valued functions on  $R$ , indexed by a well-ordered index set  $A$ ). (1) If  $\{f_\alpha\}$  is such that the sets  $V_\alpha = \{x \in R | f_\alpha(x) > 0\}$  cover  $R$ , and if  $\sup_{\beta < \alpha} f_\beta$  is continuous for all  $\alpha$ , then the covering  $\{V_\alpha\}$  has a locally finite open refinement. (2)  $R$  is paracompact if and only if for every open covering  $\{V_\alpha\}$  there exists a family  $\{f_\alpha\}$  such that  $f_\alpha(R - V_\alpha) = 0$ ,  $\sup_{\alpha \in A} f_\alpha = 1$ , and  $\sup_{\beta \in B} f_\beta$  is continuous for every  $B \subset A$ . (3)  $R$  is metrizable if and only if there exists a family  $\{f_\alpha\}$  such that  $\sup_{\beta \in B} f_\beta$  and  $\inf_{\beta \in B} f_\beta$  are continuous for every  $B \subset A$ , and such that for every  $x \in R$  and neighborhood  $U$  of  $x$  there exists an  $f_\alpha$  with  $f_\alpha(x) < \varepsilon$  and  $f_\alpha(R - U) \geq \varepsilon$  for some

$\varepsilon > 0$ . {Reviewer's note: The author asserts that the two known results in Corollaries 3 and 5 follow from Lemma 1 and Theorem 1. The reviewer was unable to see this.}

*E. Michael* (Princeton, N.J.).

**Rudin, Walter.** Homogeneity problems in the theory of Čech compactifications. Duke Math. J. 23 (1956), 409-419.

**Rudin, Walter.** Note of correction. Duke Math. J. 23 (1956), 633.

The main result is that the homogeneity of a completely regular Hausdorff space  $X$  need not imply that of  $X^* = \beta X - X$ ; the proof makes use of the continuum hypothesis (CH). (A space is called homogeneous if for any two points, there is a homeomorphism taking one to the other.) Let  $N$  denote the discrete space of positive integers. It is shown (Theorem 4.4, CH) that  $N^*$  is not homogeneous. More generally (Theorem 4.5, CH), if  $X$  is any normal locally compact Hausdorff space containing a closed copy of  $N$ , then  $X^*$  is not homogeneous. The development is based upon some interesting properties of  $N$  and  $N^*$  that are obtained in the paper. In § I, the author discusses ultrafilters on  $N$ . For added interest, he includes a simple proof of the fact that  $N$  has  $2^c$  ultrafilters — a special case of Pospíšil's theorem [Ann. of Math. (2) 38 (1937), 845-846]; {the reviewer observes that the main idea is the same as that in Hausdorff's general proof [Studia Math. 6 (1936), 18-19]}. Further results about  $N$  include the following. Let a permutation  $\pi$  of  $N$  also be regarded as a mapping on the set of all ultrafilters, in the natural way. 1.5 (credited to H. Kenyon). If two free ultrafilters  $\Omega_1$  and  $\Omega_2$  are isomorphic as partially ordered sets (under set inclusion), then there exists  $\pi$  such that  $\Omega_2 = \pi(\Omega_1)$ ; hence there are  $2^c$  isomorphism classes, and each has exactly  $c$  members. 1.6. If  $\Omega_1 \neq \Omega_2$ , then there exists  $\pi$  such that  $\pi(\Omega_1) = \Omega_1$ ,  $\pi(\Omega_2) \neq \Omega_2$ . 1.7. The group of all permutations of  $N$  has  $2^c$  subgroups [but only two nontrivial normal subgroups; see Schreier and Ulam, Studia Math. 4 (1933), 134-141]. § II is devoted to a description of  $\beta N$  as the space of all ultrafilters on  $N$ . § III deals with  $N^*$ . 3.2.  $N^*$  has exactly  $c$  open-closed sets, and, given any two of them, there is a homeomorphism of  $\beta N$  that carries one onto the other. Since  $N^*$  is zero-dimensional, it follows that for every point  $p$  and open set  $U$ , there is a homeomorphism of  $N^*$  taking  $p$  to a point of  $U$ . 3.3. Every nonempty  $G_\delta$ -set in  $N^*$  has a nonempty interior. § IV considers consequences of the continuum hypothesis. A point  $p$  of a space is called a  $P$ -point if every  $G_\delta$ -set containing  $p$  contains  $p$  in its interior [Gillman and Henriksen, Trans. Amer. Math. Soc. 77 (1954), 340-362; MR 16, 156]. 4.2 (CH).  $N^*$  has  $2^c$   $P$ -points, and the set of all  $P$ -points is dense. 4.4 and 4.5 are quoted above. 4.7 (CH). For any two  $P$ -points of  $N^*$ , there is a homeomorphism of  $N^*$  that carries one to the other; consequently,  $N^*$  has precisely  $2^c$  homeomorphisms.

The note corrects an inaccuracy in Theorem 4.5 (pointed out by the reviewer): The Tychonoff plank is a counterexample to the result as stated in the paper, but the theorem is retrievable by the simple expedient of including normality of  $X$  in the hypothesis (as in the quotation above). *L. Gillman* (Lafayette, Ind.).

**Kolmogorov, A. N.** On certain asymptotic characteristics of completely bounded metric spaces. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 385-388. (Russian)

Ist  $X$  eine endliche Menge von  $N$  Elementen, so stellt

$\log_2 N = \log N$  ein brauchbares Mass für diejenige Information dar, die im Aufzeigen eines Elementes aus  $X$  liegt. Bei unendlichem  $X$  wird man entsprechend die angenäherte Bestimmung eines Elementes behandeln. Dazu definiert Verfasser in einem beliebigen metrischen Raum  $X$  die Ausdrücke  $N^a(\varepsilon)$ ,  $N^b(\varepsilon)$ ,  $N^c(\varepsilon)$ . Und zwar ist  $N^a$  die kleinste Mächtigkeit aller  $\varepsilon$ -Netze in  $X$  (jedes  $x \in X$  hat von einem geeigneten Netzknoten einen Abstand  $\leq \varepsilon$ ); weiter  $N^b$  die kleinste Mächtigkeit aller  $X$  überdeckenden Systeme, die aus Teilmengen von Durchmesser  $\leq \varepsilon$  bestehen; schliesslich  $N^c$  die obere Grenze der Mächtigkeiten aller Teilmengen von  $X$ , bei denen je zwei Punkte einen Abstand  $> \varepsilon$  voneinander besitzen. Satz 1 besagt

$$N^a(\varepsilon) \leq N^b(\varepsilon) \leq N^c(\varepsilon) \leq N^a(\varepsilon/2).$$

Bei totalbeschränktem  $X$  sind somit  $N^a$ ,  $N^b$ ,  $N^c$  alle endlich ( $\varepsilon > 0$ ). In der Informationstheorie interessiert besonders das asymptotische Verhalten der Logarithmen dieser Ausdrücke für  $\varepsilon \rightarrow 0$  (teilweise auch der Ausdrücke selber und der iterierten Logarithmen). Verfasser bestimmt die Asymptotik mehr oder weniger genau für totalbeschränkte Teile geläufiger Funktionenräume, wobei  $N^a$ ,  $N^b$ ,  $N^c$  dieselben Resultate ergeben. Satz 2-4 behandeln die Abhängigkeit von  $N^a$  usw. vom zugrundegelegten Raum (Bildung von Vereinigung, Durchschnitt, Potenz).

K. Zeller (Tübingen).

de Groot, J. Non-archimedean metrics in topology. Proc. Amer. Math. Soc. 7 (1956), 948-953.

A non-archimedean metric is one which satisfies  $\rho(x, y) \leq \max[\rho(x, z), \rho(y, z)]$  instead of the triangle axiom. It is shown that a topological space is archimedeanly metrizable if and only if it is metrizable and strongly 0-dimensional or, equivalently, if and only if it has an open base which is the union of a countable collection of locally finite families of open and closed sets. The proof involves an embedding in a "generalized non-archimedean Hilbert space".

P. A. Smith (New York, N.Y.).

Kasahara, Shouro. Note on the Lebesgue property in uniform spaces. II. Proc. Japan Acad. 32 (1956), 248-253.

Continuing earlier investigations [see MR 17, 389], the author here studies uniform spaces for which every countable covering has the Lebesgue property. Among other things, it is proved that a uniform space  $E$  is of this type if and only if  $E$  is countably paracompact and normal and every continuous map from  $E$  to any uniform space with a countable dense subset is uniformly continuous. This result is used to study the completion of a uniform space.

E. Michael (Princeton, N.J.).

McAuley, Louis F. A relation between perfect separability, completeness, and normality in semi-metric spaces. Pacific J. Math. 6 (1956), 315-326.

A semi-metric on a set  $S$  is a nonnegative real-valued function on  $S \times S$  such that for all  $a, b$  in  $S$ ,  $d(a, b) = d(b, a)$ , and  $d(a, b) = 0$  if and only if  $a = b$ . A topological space  $S$  is said to be semi-metrizable if there exists a semi-metric  $d$  on  $S$  preserving its topology (i.e.,  $\phi$  is a limit point of a subset  $M$  of  $S$  if and only if  $0 = d(\phi, M) = \inf_{m \in M} d(\phi, m)$ ). A semi-metric space is a semi-metrizable space together with a particular semi-metric preserving its topology. A semi-metric space  $S$  with semi-metric  $d$  is called weakly (resp. strongly) complete if for any descending sequence  $M_i$  of closed subsets such that for each  $i$ , there is a  $\phi_i$  in  $M_i$  (resp. in  $S$ ) such that

$d(\phi_i, M_i) < 1/i$ , we have  $\cap M_i$  nonempty. A semi-metrizable space is weakly (resp. strongly) complete if there is a semi-metric  $d$  on  $S$  preserving its topology such that the associated semi-metric space is weakly (resp. strongly) complete.

The author shows that every regular, hereditarily separable (semi-metrizable) strongly complete space is metrizable, and for regular semi-metric spaces, weak completeness and Cauchy completeness (defined in the natural way) coincide. He gives an example of a paracompact, completely normal, hereditarily separable, semi-metrizable space that is weakly complete but is not a Moore space, and an example of a complete (completely regular, nonnormal) Moore space that is not strongly complete. (Every Moore space is semi-metrizable, and every normal semi-metrizable space is completely normal.) Next, various kinds of separability of semi-metric spaces are discussed, and a problem of Wilson [Amer. J. Math. 53 (1931), 361-373, p. 366] concerning a class of semi-metric spaces that are metrizable is solved. Finally, the author gives three conditions  $A, B, C$  (too complicated to be stated here) such that a topological space is semi-metrizable (resp. a Moore space, resp. metrizable) if and only if it satisfies  $A$  (resp.  $A$  and  $B$ , resp.  $A, B$ , and  $C$ ).

(Reviewer's comments: (1) The author does not use the term semi-metrizable (or metrizable); it is introduced above for clarity. (2) The proof of Theorem 3.2. becomes more transparent if the reader recalls Bing's result that every collectionwise normal Moore space is metrizable [Canad. J. Math. 3 (1951), 175-186; MR 13, 264]. (3) The reviewer found that  $A$  became clearer on replacing " $g(\phi)$ " by " $g_n(\phi)$ " in (b), and " $g_i(\phi_i)$ " by " $g_n(\phi_i)$ " in (c).)

M. Henriksen (Princeton, N.J.).

Anderson, R. D. One-dimensional continuous curves. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 760-762.

These theorems are stated; proofs will be given elsewhere.

Let  $\mathfrak{K}$  be a collection of one-dimensional locally connected compact metric continua, no two being homeomorphic to each other and each such continuum being homeomorphic to some element of  $\mathfrak{K}$ . Let  $\mathfrak{M}$  be the set of elements of  $\mathfrak{K}$  not containing any local cut points.

Theorem I. If  $M$  is an element of  $\mathfrak{M}$ , then in order that  $M$  be homeomorphic to Menger's universal curve, it is necessary and sufficient that  $M$  contain no open subset imbeddable in the plane.

Theorems II, III. If  $K$  is an element of  $\mathfrak{K}$ , then in order that  $K$  be the universal curve, it is necessary and sufficient that  $K$  be cross-connected, and in order that  $K$  be homogeneous, it is necessary and sufficient that  $K$  be either the universal curve or the simple closed curve.

Theorem IV. Any zero-dimensional compact metric group  $G$  can operate on the universal curve (as a continuous transformation group) in a fixed-point-free fashion; the orbit space may be required to be the universal curve, or, if  $G$  is infinite, to be a regular curve.

There are two further theorems on collapsed mappings.

H. M. Gehman (Buffalo, N.Y.).

Michael, Ernest. Continuous selections. II. Ann. of Math. (2) 64 (1956), 562-580.

This is the second of a series of papers on continuous selections [for motivation and terminology not defined below, see Michael, Ann. of Math. (2) 63 (1956), 361-382; MR 17, 990]. A topological space  $Y$  is said to be  $C^*$  if every continuous mapping of the  $m$ -sphere ( $m \leq n$ ) into  $Y$

has a continuous extension over the  $(m+1)$ -ball into  $Y$ . A family  $SC^2^Y$  is equi- $LC^n$  if for every  $y$  in some member of  $\mathcal{S}$ , and every neighborhood  $U$  of  $y$  in  $Y$ , there is a neighborhood  $V$  of  $u$  in  $Y$  such that for every  $S \in \mathcal{S}$ , every continuous mapping of the  $m$ -sphere ( $m \leq n$ ) into  $S \cap V$  has a continuous extension over the  $(m+1)$ -ball into  $S \cap U$ . If  $A$  is a closed subset of  $Y$ , then  $\dim_Y(Y-A) \leq n$  means that the Lebesgue dimension of every closed subset of  $Y$  contained in  $Y-A$  is less than or equal to  $n$ .

The main theorem of the paper is the following: Let  $X$  be a paracompact space, and let  $A$  denote a closed subset of  $X$  with  $\dim_X(X-A) \leq n+1$ ; let  $Y$  be a complete metric space, let  $\mathcal{S}$  be an equi- $LC^n$  family of closed non-empty subsets of  $Y$ , and let  $\varphi: X \rightarrow \mathcal{S}$  be lower semi-continuous. Then every selection for  $\varphi|A$  can be extended to a selection for  $\varphi|U$  for some open  $U \supset A$ . If also every  $S \in \mathcal{S}$  is  $C^n$ , then one can take  $U=X$ . Several applications of this theorem are given, and more are promised in the third paper of the series. Moreover, a partial converse of the theorem is given to show why it is necessary to assume that  $\mathcal{S}$  is equi- $LC^n$ . A number of the intermediate results in the paper are of independent interest.

{Reviewer's remark: The author has requested me to note that the definitions given in the paper of  $LC^n$ ,  $C^n$ , equi- $LC^n$ , uniformly  $LC^n$ , and uniformly equi- $LC^n$  are inaccurate. Correct definitions of two of them are given above from which correct versions of the others may easily be derived.}

M. Henriksen (Princeton, N.J.).

Wintner, Aurel. Sur le dernier théorème de géométrie de Poincaré. C. R. Acad. Sci. Paris 243 (1956), 835-836.

The following generalization of the Poincaré-Birkhoff fixed-point theorem is given. Consider the class of continuously differentiable homeomorphisms of the annulus which transform the boundary curves as in the Poincaré-Birkhoff theorem. Then given  $\epsilon > 0$  there is a  $\delta > 0$  such that if any  $\beta$  has  $|j(P) - 1| < \delta$  for all  $P$ , where  $j$  is the Jacobian determinant of  $\beta$ , then  $|P - \beta(P)| < \epsilon$  for some  $P$ .

E. E. Floyd (Charlottesville, Va.).

Fort, M. K., Jr. The embedding of homeomorphisms in flows. Proc. Amer. Math. Soc. 6 (1955), 960-967.

Let  $f$  be a homeomorphism of a topological space  $X$  onto itself and let  $\{F_t\}: X \times \{t \mid t \text{ real}\} \rightarrow X$  be a continuous flow in  $X$ . We say that  $f$  is embedded in  $\{F_t\}$  if  $F_1 = f$ .

The author restricts himself to the case where  $X$  is an interval on the real line. For this case he gives conditions under which a given  $f$  is embeddable in some continuous flow  $\{F_t\}$ . E.g. Theorem 1. A necessary and sufficient condition that a homeomorphism  $f$  of an interval onto itself is embeddable in a continuous flow is that  $f$  is order preserving. Theorem 2: If  $f, f(x) > x, a < x < b$ , is a continuously differentiable homeomorphism of  $(a, b]$  onto  $(a, b]$  with a positive, monotone derivative  $f'$ , then there exists a unique flow  $\{F_t\}$  on  $(a, b]$  which embeds  $f$  and is such that for every  $t, F_t$  is continuously differentiable on  $(a, b]$ . Moreover if  $g$  is any continuously differentiable homeomorphism of  $(a, b]$  onto  $(a, b]$  which commutes with  $f$  then  $g = F_s$  for some real number  $s$ .

Under mildly more restrictive conditions the author proves that the totality  $G$  of homeomorphisms  $g$  mentioned in the theorem 2 is a group which is isomorphic either to the groups of integers or to the group of reals.

An example is given to show that theorem 2 is not true if one changes  $(a, b]$  to  $[a, b]$ . Y. N. Dowker (London).

Ford, G. W.; and Uhlenbeck, G. E. Combinatorial problems in the theory of graphs. III. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 529-535.

The author discuss the asymptotic behaviour of solutions to counting problems for graphs given in two earlier papers [Ford and Uhlenbeck, same Proc. 42 (1956), 122-128; Ford, Norman, and Uhlenbeck, ibid. 42 (1956), 203-208; MR 17, 1231] for graphs in which the number of points is large.  $T(p)$  is the number of rooted mixed star trees with  $p$  labelled points which are built out of stars from a given collection of types of star.  $T(x) = \sum_{p=1}^{\infty} T(p)x^p/p!$  has radius of convergence  $x_0$  and in the neighbourhood of  $x_0$  may be expanded in powers of  $(x-x_0)^{1/2}$  in the form  $T(x) = T(x_0) - b(x-x_0)^{1/2} + \dots$ .  $d$  is the greatest common divisor of the set of integers  $q_\sigma - 1$ , where  $q_\sigma$  is the number of points of a star of type  $\sigma$  and  $\sigma$  ranges over all types of star in the collection. Then

$$\frac{1}{p!} T(p) \sim \begin{cases} 0, & \frac{p-1}{d} \text{ not integer,} \\ d \frac{b}{2\pi i} x_0^{-p+1/2} p^{-3/2}, & \frac{p-1}{d} \text{ an integer.} \end{cases}$$

An analogous result for the number of star trees with unlabelled points is also obtained. G. A. Dirac.

Vitalbi, Luciano. Ricerche sulla teoria dei reticoli (sistemi "parzialmente ordinati" nei quali vi è un ordine di "precedenza" in ogni stadio). Giorn. Mat. Battaglini (5) 4(84) (1956), 93-121 (15 plates).

A more descriptive title would be "A relationship between trees and expressions". Indeed, the word "reticolo" (lattice) appears only in the title and first sentence, never thereafter. The author follows Andreoli [same Giorn. (5) 2(82) (1954), 237-266; MR 15, 892] in defining a tree as a finite set of points (nodes) and line segments (branches), each branch connecting two nodes, containing no closed circuit; [this definition does not assure connectedness, as the author apparently assumes]. A tree is "directed" if one node is chosen as origin (root) and the other nodes are partially ordered by proceeding successively outward to the terminal nodes. An algebraic expression can be represented as a tree where the branches are the successive partial expressions whose combination by the permitted operations eventually yield the given expression, and where the operations are represented by the nodes of the tree (except that the basic constants or free variables (at the terminal points) and the given expression seem to need special treatment which is not clearly stated); thus addition may be represented by each of several nodes. This idea, and numerous examples of it for the cases of rational functions and functions involving square roots, are examined in leisurely detail. Also discussed is the relationship of this representation to the dual representation in which operations correspond to branches and expressions to nodes. Briefer consideration is given to similar representations of the derivative (of a sum, product, etc.), logarithm, and so on. The bibliography is scanty.

P. M. Whitman.

See also: Mikulik, p. 275; Lesieur, p. 275; Guérindon, p. 289; Polak, p. 289; Kaufmann, p. 290; Položil, p. 291; Ohtsuka, p. 292; Kawakami, p. 295; Choquet, p. 295; Brelot, p. 296; Klee, p. 330; Špaček, p. 330; Schröder, p. 337.

# Algebraic Topology

**Bokštejn, M. F.** Duality theorem for locally bicomact spaces. *Moskov. Gos. Univ. Uč. Zap.* 145, Mat. 3 (1949), 131-164. (Russian)

The author gives a complete, and very detailed, proof of the Alexander-Kolmogorov duality theorem. If  $R$  is locally compact Hausdorff,  $A$  a closed subset,  $H^q(R) = H^{q+1}(R) = 0$ , the  $H^q(A) \approx H^{q+1}(R-A)$ .  $R$  is not assumed normal. The cohomology groups  $H^q$  are based on finite open coverings in the sense of Aleksandrov [finite set of pairwise different open sets with  $R$  as union, closed under intersection]. They are essentially groups with compact carriers: the value of a co-chain on a simplex of the nerve is 0, if one of the vertices is a set whose closure is not compact. The main new auxiliary concept is that of "covering regular with respect to  $A$ ", meaning that any set  $v$  in the covering with compact closure  $\bar{v}$  and with  $v \cap A = \emptyset$  has  $\bar{v} \cap A = \emptyset$ .

H. Samelson.

**Berikašvili, N. A.** On the homology groups of a space with a compact coefficient group. *Soobšč. Akad. Nauk Gruzin. SSR* 16 (1955), 753-760. (Russian)

The author considers compact direct sums and limits of compact groups, following Čogošvili [Mat. Sb. N.S. 28(70) (1951), 89-118; MR 12, 846]. He then defines compact homology groups, with compact coefficients, for (infinite) simplicial complexes, and verifies the Eilenberg-Steenrod axioms in the simplicial form. For an arbitrary pair  $(X, A)$  ( $X$  a space,  $A$  a subset) he defines the homology groups as compact direct limits of the groups of the nerves of (infinite) coverings. These groups turn out to be the duals of the Čech cohomology groups, based on infinite coverings, with the (discrete) dual group for coefficients.

H. Samelson (Ann Arbor, Mich.).

**Grothendieck, A.** Théorèmes de finitude pour la cohomologie des faisceaux. *Bull. Soc. Math. France* 84 (1956), 1-7.

If  $X$  is a compact complex manifold,  $N$  a coherent analytic sheaf (faisceau) on  $X$ , then the cohomology groups  $H^q(X, N)$  are finite-dimensional vector spaces [Cartan and Serre, C. R. Acad. Sci. Paris 237 (1953), 128-130; MR 16, 517]. In this paper the author gives a number of general theorems of which the preceding is an important special case. Moreover the author's proof is considerably simpler than the original proof of Cartan-Serre.

A sheaf  $N$  (on a paracompact space  $X$ ) is said to be calculable up to dimension  $p$  if, for each  $x \in X$ , and every neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V \subset U$  of  $x$  such that the natural map  $H^q(U, N) \rightarrow H^q(V, N)$  is zero for  $1 \leq q \leq p$ . The author proves that if  $N$  is calculable up to dimension  $p$  then the cohomology groups  $H^q(X, N)$  for  $q \leq p$  are calculable in the following sense. There exists a pair of open coverings  $U, V$  (arbitrarily fine and locally finite) with  $V$  a refinement of  $U$ , such that  $H^q(X, N)$  can be identified with the quotient of  $H^q(C(U, N))$  by the kernel of  $H^q(C(U, N)) \rightarrow H^q(C(V, N))$ , where  $C(U, N)$  is the cochain complex of the covering  $U$  with coefficients in  $N$ .

A sheaf  $N$  of vector spaces (real or complex) on  $X$  is said to be a topological vector sheaf (faisceau vectoriel topologique) if (i) for each open set  $U$  the space of sections  $\Gamma(U, N)$  of  $N$  over  $U$  is a topological vector space, (ii) for each open covering  $\{U_i\}$  of  $U$  the topology of  $\Gamma(U, N)$  given in (i) is the coarsest topology for which all the maps  $\Gamma(U, N) \rightarrow \Gamma(U_i, N)$  are continuous.

$N$  is said to be compact if, for each open set  $V$  relatively compact in an open set  $U$ , the map  $\Gamma(U, N) \rightarrow \Gamma(V, N)$  is compact. The main theorem is then as follows: Let  $N$  be a compact sheaf on a compact space  $X$ , and suppose  $N$  is calculable up to dimension  $p$ , then  $H^p(X, N)$  is finite-dimensional. Since  $N$  is calculable up to dimension  $p$ ,  $H^p(X, N)$  is calculable and, a suitable pair of finite coverings  $U, V$  being chosen as above, the theorem reduces to a compactness theorem of L. Schwarz on spaces of type (F) [ibid. 236 (1953), 2472-2473; MR 15, 233]. The theorem of Cartan-Serre follows as a corollary; it is only necessary to know Theorems "A" and "B" for polydiscs.

Finally, a more general result is proved ( $X$  being now a compact metrisable space) in which the condition that  $N$  be calculable is replaced by another condition assuming the existence of a suitable resolution of  $N$  by topological vector sheaves. This result applies in particular to elliptic differential operators on a compact indefinitely differentiable manifold.

On page 6, 4 lines from the bottom, "enfin  $G$  fini" should read "enfin  $G$  fin".

M. F. Atiyah (Cambridge, England).

**Al'ber, S. I.** Dual homological sequences. *Dokl. Akad. Nauk SSSR (N.S.)* 108 (1956), 763-766. (Russian)

The author extends a method which he has used already in several particular cases [same Dokl. (N.S.) 91 (1953), 1237-1240; 98 (1954), 325-328; MR 15, 457; 17, 71] for easily computing the homology of certain types of manifolds. Let  $A$  and  $B$  be intersecting subsets of an oriented Manifold  $M^n$  such that  $M^n/A$  is homotopically equivalent to  $B$  and  $M^n/B$  to  $A$  and let  $\Gamma, \Phi$  be dual coefficient groups. The author's method is based on the remark that, in the exact homology sequences  $(M^n, A, \Gamma)$  and  $(M^n, B, \Phi)$ , the terms of dimension  $p$  of one sequence are dual to the appropriate terms of dimension  $q = n - p$  of the other — {The statement of this remark contains a couple of misprints,  $p$  having been written for  $q$ .}

L. C. Young (Madison, Wis.).

**Whitehead, George W.** Homotopy groups of joins and unions. *Trans. Amer. Math. Soc.* 83 (1956), 55-69.

The purpose of this paper is to study the homotopy groups of the join of two spaces  $X$  and  $Y$ , denoted by  $X * Y$ . The main result is contained in the following theorem: Let  $X$  be an  $(m-1)$ -connected space and  $Y$  an  $(n-1)$ -connected CW-complex ( $m, n > 1$ ). Then there exists a spectral sequence  $\{E^r\}$  of bi-graded groups such that  $E_{p,q}^2$  is isomorphic to the reduced homology group  $H_{n+p}(Y; \pi_{m+q}(X))$  for  $q < m-1$ , and  $E^\infty$  is the graded group associated with the group  $\sum_r \pi_r(X * Y)$  with respect to a suitable filtration.

This spectral sequence is obtained as follows. Let  $Y^k$  denote the  $k$ -dimensional skeleton of  $Y$ . Then the sets  $X * Y^k$  for  $k=0, 1, 2, \dots$  form a "filtration" of the space  $X * Y$ . By considering the exact homotopy sequences of all the pairs  $(X * Y^k, X * Y^{k-1})$  simultaneously, one obtains an exact couple. The spectral sequence associated with this exact couple is the desired spectral sequence.

The author gives two applications of this theorem. For the first application, he takes  $X$  and  $Y$  to be Eilenberg-MacLane spaces,  $X = K(G, m)$ , and  $Y = K(\Pi, n)$ , where  $G$  and  $\Pi$  are arbitrary abelian groups. In this case the spectral sequence simplifies somewhat due to the fact that so many of the groups involved are trivial. By making use of the commutativity of the join  $(X * Y =$

$Y \cdot X$ ), the following result is obtained: the "stable" Eilenberg-MacLane homology groups  $H_{p+q}(\Pi, p; G)$  and  $H_{p+q}(G, p; \Pi)$ ,  $p > q$ , are isomorphic. This result was previously proved by H. Cartan [Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 704-707; MR 16, 390] by completely different methods.

The second application of the above-mentioned theorem is to the study of the homotopy groups of the union  $X \vee Y$  of the spaces  $X$  and  $Y$  with a single point in common. It is well-known that the homotopy group  $\pi_n(X \vee Y)$  is naturally isomorphic to the direct sum of the following groups:  $\pi_n(X)$ ,  $\pi_n(Y)$ , and  $\pi_{n+1}(X \times Y, X \vee Y)$ . The author defines a homomorphism of  $\pi_{n+1}(X \times Y, X \vee Y)$  into  $\pi_{n+2}(X * Y)$  which is an isomorphism in low dimensions. Thus any results on the homotopy groups of  $X * Y$  can be translated into results on the homotopy groups of  $X \vee Y$ . W. S. Massey (Providence, R.I.).

**Kobayashi, Shoshichi.** Principal fibre bundles with the 1-dimensional toroidal group. Tôhoku Math. J. (2) 8 (1956), 29-45.

The set of all principal fiber bundles over a manifold  $M$  with the circle group  $T^1$  as the structural group forms an additive group  $P(M, T^1)$ . By taking the characteristic class of the bundle, one defines a homomorphism  $P(M, T^1) \rightarrow H^2(M, Z)$ , the latter being the second co-

homology group of  $M$  with the integers  $Z$  as coefficients. This homomorphism is studied from three different viewpoints: 1) differential-geometric, with the introduction of a connection; 2) sheaf-theoretic; 3) homotopy. The results are applied to bundles with the unitary group, submanifolds in Hermitian space, and homogeneous manifolds. In particular, the author determines the group  $P(G/K, T^1)$ , where  $G$  is a connected and simply connected Lie group and  $K$  is a connected compact subgroup of  $G$ . S. Chern (Chicago, Ill.).

**Brahana, Thomas R.** Products of quasi-complexes. Proc. Amer. Math. Soc. 7 (1956), 954-958.

Lefschetz [Algebraic topology, Amer. Math. Soc. Colloq. Publ., v. 27, New York, 1942; MR 4, 84] introduced the notion of a quasi-complex, and proved that his fixed point theorem held on such spaces. It is here proved that the cartesian product of two quasi-complexes is a quasi-complex. This answers a question asked by Dyer at the Summer Institute on Set Theoretic Topology. E. E. Floyd (Charlottesville, Va.).

See also: Skopin, p. 276; Eichler, p. 297; Gagaev, p. 301; Haas, p. 305; Kobayashi, p. 332; Memoirs of the unifying study... p. 332; Kashiwabara, p. 332; Kreyszig, p. 333; Takasu, p. 363.

## GEOMETRY

### Geometries, Euclidean and Other

**Scott, Dana.** A symmetric primitive notion for Euclidean geometry. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 456-461.

Pieri [Mem. Mat. Fis. Soc. Ital. Sci. (3) 15 (1908), 345-450] proved that the relation  $I(x, y, z) = (x \text{ is equidistant from } y \text{ and } z)$  is sufficient as the only primitive notion for euclidean geometry in any number of dimensions  $\geq 2$ . Using this result, the author proves by elementary methods, that the relations  $S(x, y, z) = (x, y, z \text{ form a non-degenerate right triangle})$  suffices as well. A. Heyting.

**Beth, Evert W.; and Tarski, Alfred.** Equilaterality as the only primitive notion of Euclidean geometry. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 462-467.

Using Pieri's result [see the preceding review] the authors prove that the relation  $E(x, y, z) = (x, y, z \text{ form an equilateral triangle or they all coincide})$  is sufficient as the only primitive notion for euclidean  $n$ -dimensional geometry, provided  $n \geq 3$ . If  $n=2$ ,  $E$  cannot serve as the only primitive notion. To prove the latter, a one-to-one transformation  $T$  of the euclidean plane onto itself is constructed, which leaves  $E$  invariant, but not Pieri's relation  $I$ . This construction involves the axiom of choice; it is highly improbable that the result can be obtained without its use. A. Heyting (Amsterdam).

**Tarski, Alfred.** A general theorem concerning primitive notions of Euclidean geometry. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 468-474.

Proof of the following theorem: Let  $R$  be a ternary relation between points of the euclidean plane (identified with complex numbers) which can serve as the only primitive notion for plane euclidean geometry. Let  $a_1$  and  $a_2$  be any two distinct complex numbers, and  $C$  the

set of all complex numbers  $c$  for which  $R(a_1, a_2, c)$  holds. Then the number field  $\bar{C}$ , generated by  $a_1, a_2$  and the elements of  $C$ , contains all complex numbers. The method is analogous to that by which Beth and Tarski [see the preceding review] derived the negative part of their result. This and Pieri's result that  $I(x, y, z)$  [see the second preceding review] cannot serve as the only primitive notion of one dimensional euclidean geometry, are special cases of the theorem. A. Heyting (Amsterdam).

**Teodorescu, Ioana.** Collineation of three points. Gaz. Mat. Fiz. Ser. A. 8 (1956), 266-271. (Romanian)

Some remarks about the methods by which collinearity may be proved in elementary plane geometry. One example is:  $ABCD$  is a quadrangle,  $ABE$  and  $BCF$  are equilateral triangles,  $E$  is inside and  $F$  outside the quadrangle,  $D, E$  and  $F$  are collinear. The theorems of Simson, Menelaos and Gauss. O. Bottema (Delft).

**Steiner, H. G.** Bewegungsgeometrische Lösung einer Dreiecks-konstruktion. Math.-Phys. Semesterber. 5 (1956), 132-137.

A triangle  $P_1P_2P_3$  is given. The problem is to construct a triangle  $ABC$  so that the three triangles  $P_1AB, P_2BC, P_3CA$  shall all be equilateral. There are, in general, eight solutions. H. S. M. Coxeter (Toronto, Ont.).

**Avdis, J.; et Thébault, V.** Sur la géométrie du tétraèdre. Mathesis 65 (1956), 214-218.

**Veldkamp, G. R.** A theorem from elementary plane geometry. Nieuw Tijdschr. Wisk. 44 (1956/57), 1-4. (Dutch)

Very short proof of Pompeu's theorem: If  $A_1A_2A_3$  is an equilateral triangle and  $P$  an arbitrary point in its plane, then there is a triangle with  $PA_i$  as its sides. The theorem is also valid if  $P$  is outside the plane.

O. Bottema (Delft).

**Thébault, Victor.** Sur la droite de Simson. *Mathesis* 65 (1956), 201-205.

Theorems concerning the Simson line in special position, e.g. passing through the center of the circumscribed circle.

**Stone, A. P.** On the stereographic projection of the sphere. *Math. Gaz.* 40 (1956), 181-184.

Using analytical expressions for stereographic projection, the author gives new proofs for some classical formulae of spherical trigonometry. The construction used on astrolabes for determining planetary time is explained by regarding the plane figure as a stereographic projection of a simpler figure on a sphere.

*H. S. M. Coxeter* (Toronto, Ont.).

★ **Klein, Felix.** Lectures on the icosahedron and the solution of equations of the fifth degree. Translated into English by George Gavin Morrice. Second and revised edition. Dover Publications, Inc., New York, N.Y., 1956. xvi+289 pp. \$1.85.

A republication of the English translation of the second revised edition [Kegan-Paul, London, 1914].

**Matsuno, Takeshi.** On star-like theorems and convex-like theorems in the complex vector space. *Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A.* 5 (1955), 88-95.

Denote by  $Z$  the  $n$ -dimensional column vector with elements  $z_1, \dots, z_n$ , and denote by  $W(Z)$  a similar vector with elements  $w_1(Z), \dots, w_n(Z)$ . Denote by  $D_r$  the image of  $\|Z\| < r$  by the mapping  $W=W(Z)$ , and let  $(X, Y)^*$  be the angle between the vectors  $X$  and  $Y$ , defined by the equation

$$\cos(X, Y)^* = \frac{\operatorname{Re}(X^*Y)}{\|X\| \|Y\|}, \quad 0 \leq (X, Y)^* < 2\pi.$$

Finally denote by  $N_W$  the outer normal vector of  $D_r$  at the point  $W$  on the boundary of  $D_r$ . The authors consider regular functions  $W=W(Z)$  in  $\|Z\| < 1$  with  $W(0)=0$ . They say that  $W(Z)$  is starlike (with respect to the origin) in  $\|Z\| < 1$  when  $(N_W, W)^* < \frac{1}{2}\pi$  for  $W$  on the boundary of  $D_r$  and  $0 < r < 1$ . They prove the following theorem. Let  $W=Z$  + higher powers be regular in  $\|Z\| < 1$  with non-vanishing jacobian  $J = |\det dW/dZ|^2$ . Then a necessary and sufficient condition for the univalence and the starlikeness of  $W(Z)$  in  $\|Z\| < 1$  is that  $\operatorname{Re}\{W^*(dZ/dW)^*Z\} > 0$  for  $0 < \|Z\| < 1$ . They also obtain a theorem on the number of zero points of  $W(Z)$  in  $\|Z\| < r$ , and a theorem giving a necessary and sufficient condition for the conformality of the mapping  $W=W(Z)$ .

*W. T. Martin.*

**Kuiper, Nicolaas H.** Eine charakteristische Eigenschaft der Kurven zweiter Ordnung. *Math.-Phys. Semesterber.* 5 (1956), 138-140.

Strengthening a result of W. Süss [same *Semesterber.* 4 (1954), 54-56; *MR* 16, 279], the author proves the following theorem. If a system of closed curves in the real affine plane includes, with each member, all affinely related curves, and if the number of intersections of two members never exceeds four, then the system consists of ellipses.

He also proves a slightly more complicated theorem about projectively related curves in the real projective plane.

*H. S. M. Coxeter* (Toronto, Ont.).

**Tiago de Oliveira, J.** The theory of modules and the construction of linear space. *Ciência, Lisboa* 2 (1956), no. 13, 25-34. (Portuguese)

**Facciotti, G.** Piani non desarguesiani. *Period. Mat.* (4) 34 (1956), 159-168.

**Medek, Václav.** Linear systems of projective transformations on a straight line. *Mat.-Fyz. Časopis. Slovensk. Akad. Vied* 6 (1956), 98-108. (Slovakian. Russian summary)

The projectivities on the real projective straight line  $P_1$  in homogeneous coordinates  $x_1, x_2$  may be represented by  $2 \times 2$ -matrices  $A = (a_{ij})$  ( $i, j = 1, 2$ ) of rank  $\geq 1$ . They are thus in one-one correspondence with the points of real projective space  $P_3$  if the four real numbers  $a_{ij}$  are taken as homogeneous coordinates in  $P$ . The points of the quadric  $Q$  in  $P_3$ , given by the equation  $|A| = a_{11}a_{22} - a_{12}a_{21} = 0$  (a hyperbolic paraboloid) represent the "singular projectivities"  $A$ . All projectivities on a line in  $P_3$ , viz.  $\lambda_1 A + \lambda_2 B$  ( $A, B$  linearly independent) form a pencil, those on a plane: a bundle. The point  $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  represents

the identity. The fundamental figure for the purpose of classifying all the real one-dimensional projectivities consists of the following parts: The quadric  $Q$ , the point  $E$ , its polar plane  $\varepsilon$  with respect to  $Q$ , the cone  $K$  generated by all tangents to  $Q$  through  $E$ , the conic section  $\varepsilon = Q \cap \varepsilon = Q \cap K$ , and the family of quadrics  $Q_\lambda$  obtained from  $Q$  by perspective collineation with  $E$  as center and  $\varepsilon$  as invariant plane. A projective coordinate substitution  $x = B\bar{x}$  ( $|B| \neq 0$ ) on  $P_1$  induces a similarity transformation of the matrices corresponding to points in  $P_3$ , viz.  $\bar{A} = B^{-1}AB$ ; evidently this transformation leaves invariant  $E$  and  $Q$  and each  $Q_\lambda$ . Thus by an adequate description of all possible situations of a line or a plane in  $P_3$  with respect to the (in all its parts invariant) fundamental figure the author is led to a projective similarity classification of all pencils or bundles of real one-dimensional projectivities. He obtains 17 types of pencils and 7 types of bundles. *H. Schwerdtfeger* (Melbourne).

**Solnceva, T. V.** Some remarks on the article of D. Z. Gordevskii "Multidimensional analogues of the hyperboloid". *Uspehi Mat. Nauk* (N.S.) 11 (1956), no. 3(69), 175-176. (Russian)

It is pointed out that most results in the article mentioned in the title [*Uspehi Mat. Nauk* (N.S.) 10 (1955), no. 3(65), 129-133; *MR* 17, 183] are false or senseless, and that adequate definitions of higher dimensional hyperboloids and their properties are already found in the classical works of Bertini and C. Segre. *H. Busemann.*

**van der Woude, W.** On the group of rotations in  $R_4$ . *Nederl. Akad. Wetensch. Proc. Ser. A.* 59=Indag. Math. 18 (1956), 367-370.

Sequel to a former paper by the author [*Ann. Scuola Norm. Sup. Pisa* (2) 4 (1935), 163-174] where the homomorphism was considered of the group  $G_1$  of rotations in six-dimensional euclidean space to a certain group  $G_2$  of Linear transformations in four-dimensional vectorspace  $R_4$ . The group  $G_2^1$  of rotations in  $R_4$  is a subgroup of  $G_2$ . Author answers the question which subgroup  $G_1^1$  of  $G_1$  is connected to  $G_2^1$  by the homomorphism.  $G_1^1$  is build up of the rotations in  $R_4$  expressed by two rotations in parallel planes.

*O. Bottema* (Delft).

See also: Andreoli, p. 275; Loewner, p. 318; Heppes und Révész, p. 323; Radziszewski, p. 330; Kobayashi, p. 332; Memoirs of the unifying study..., p. 332; Soós, p. 333; Rényi, p. 339; Plainevaux, p. 346; Burgers, p. 348; Reulas, p. 362; Elias, p. 365.

## Convex Domains, Integral Geometry

★ Bateman, P. T.; Radstrom, Hans; Hanner, Olaf; Macbeath, A. M.; Rogers, C. A.; and Klee, V. L. *Seminar on convex sets*. The Institute for Advanced Study, Princeton, N. J., 1949-1950. Reprinted November, 1955. 88 pp.

The 1949-1950 set of these mimeographed notes was reviewed in MR 16, 278.

★ Pogorelow, A. W. *Die eindeutige Bestimmung allgemeiner konvexer Flächen*. Akademie-Verlag, Berlin, 1956. 79 pp. DM 5.50.

A translation by J. Naas of the book reviewed in MR 16, 162.

Radziszewski, Konstanty. *Sur les cordes qui partagent l'aire d'un ovale en 2 parties égales*. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 8 (1954), 89-92 (1956). (Polish and Russian summaries)

Solving a problem proposed by Biernacki [same Ann. 7 (1953), 103-112; MR 16, 950], the author proves that every oval of diameter  $D$  admits a chord, of length  $> \frac{1}{2}D$ , dividing the area into two equal parts. The case of a triangle with two very small angles shows that the coefficient  $\frac{1}{2}$  cannot be improved. H. S. M. Coxeter.

Radziszewski, Konstanty. *Sur les cordes qui partagent le périmètre d'un ovale en 2 parties égales*. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 8 (1954), 93-96 (1956). (Polish and Russian summaries)

The author proves that every oval of diameter  $D$  admits a chord, of length  $> 0.829D$ , dividing the circumference into two equal parts. The critical value is obtained by considering an isosceles triangle whose equal angles  $2\phi$  satisfy the equation

$$2 \sin^3 \phi + \sin \phi - 1 = 0.$$

H. S. M. Coxeter (Toronto, Ont.).

Hadwiger, H. *Konkave Eikörperfunktionale und höhere Trägheitsmomente*. Comment. Math. Helv. 30 (1956), 285-296.

A functional  $\varphi(K)$  defined on the convex bodies in  $E^n$  is concave if  $\varphi(\lambda K + \mu L) \leq \lambda \varphi(K) + \mu \varphi(L)$ . The norm of a convex body is its average width multiplied by  $\pi^{1/2} / \Gamma(\frac{1}{2}n)$ . For a concave functional  $\varphi(K)$  which is invariant under motion of  $K$  the sphere yields the maximal value among all bodies with a given norm. For a given convex body  $K$  denote by  $S$  and  $S^0$  the spheres which have the same volume and norm as  $K$ . Put  $I_0(K) = 1$  and

$$I_r(K) = c_r^{-1} \int_K \cdots \int_K |s, p_1, \dots, p_r|^2 dp_1 \cdots dp_r \quad (1 \leq r \leq n),$$

where  $c_r = n! - 1 \binom{n}{r}$ ,  $s$  is the center of gravity of  $K$ ,  $|s, p_1, \dots, p_r|$  the volume of the  $r$ -simplex with vertices  $s, p_1, \dots, p_r$ ; and  $p_i$  range independently over  $K$ . Then

$$I_1(S) \geq I_1(K) \geq I_2(K)^{1/2} \geq \cdots \geq I_n(K)^{1/n} \geq I_n(S^0),$$

$$I_a(K)^{b-c} I_b(K)^{c-a} I_c(K)^{a-b} \geq 1 \text{ for } 0 \leq a < b < c \leq n.$$

$I_1(K)$  is, of course, the ordinary polar moment of inertia of  $K$ . The first inequality is proved by showing that  $I_1(K)^{1/(n+2)}$  is a concave functional. It was known previously only in the case  $n=2$  [see Pólya and Szegő, *Isoperimetric inequalities in mathematical physics*, Princeton, 1951; MR 13, 270]. H. Busemann (Los Angeles, Calif.).

Klee, V. L., Jr. *The structure of semispaces*. Math. Scand. 4 (1956), 54-64.

If  $L$  is a real linear space and  $p \in L$ , a semispace at  $p$  is a maximal convex subset of  $L - \{p\}$ . (Semispaces were first studied by Hammer [Duke Math. J. 22 (1955), 103-106; MR 16, 612], who showed that the class of all semispaces in  $L$  is the smallest intersection-base for the class of convex proper subsets of  $L$ .) The author's first theorem asserts that a subset of  $L$  is a semispace if and only if it can be generated in a canonical fashion by a certain kind of simply ordered set  $\mathcal{F}$  of linear functionals on  $L$ ; moreover,  $\mathcal{F}$  can always be taken as the set of coordinate functionals associated with a basis in  $L$  if and only if  $\dim L \leq \aleph_0$ . Next it is shown that if  $k$  is the number of isomorphism types represented by the semispaces of  $L$ , then  $k=1$  when  $\dim L < \aleph_0$ , and  $k=2^{\dim L}$  when  $\dim L = \aleph_0$  or  $\dim L \geq 2^{\aleph_0}$ . Finally, it is proved that if  $\dim L \leq \aleph_0$ , and  $C$  is a convex proper subset of  $L$ , then  $C$  is the intersection of a countable family of semi-spaces if and only if every family of convex sets whose intersection is  $C$  contains a countable subfamily whose intersection is  $C$ . An analogous topological result for separable  $F$ -spaces is also obtained. E. Michael (Princeton, N. J.).

Bellman, Richard. *Converses of Schwarz's inequality*. Duke Math. J. 23 (1956), 429-434.

The author proves the following theorem: If  $u(x), v(x)$  are concave in  $[0, 1]$  and  $u(0)=v(0)=u(1)=v(1)=0$ , then

$$\left[ \int_0^1 u(x)v(x)dx \right]^2 \geq \frac{1}{4} \int_0^1 u(x)^2 dx \int_0^1 v(x)^2 dx.$$

He proves also a generalization for the multidimensional case and a similar converse of the Hölder-inequality. {Besides the papers quoted by the author, the following theorem [Pólya and Szegő, *Aufgaben und Lehrsätze aus der Analysis*, Bd. I, Springer, Berlin, 1925, pp. 57, 214; Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge, 1934, p. 166] on this subject might be mentioned: If  $0 < a \leq u(x) \leq A < \infty$ ,  $0 < b \leq v(x) \leq B < \infty$ , then

$$\left[ \int_0^1 u(x)v(x)dx \right]^2 \geq 4 \left[ a^{-1}b^{-1}A^1B^1 + a^1b^1A^{-1}B^{-1} \right] \times \int_0^1 u(x)^2 dx \int_0^1 v(x)^2 dx.$$

J. Aczél (Debrecen).

Biernacki, Mieczysław. *Sur quelques propriétés des fonctions de distances. II*. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 8 (1954), 81-88 (1956). (Polish and Russian summaries)

[For part I see J. Math. Pures Appl. (9) 31 (1952), 305-318; MR 14, 679.] Let  $D_1, D_2, \dots, D_n$  denote  $n$  lines, and  $E$  a bounded, closed subset of a plane. If  $d_i(p)$  is the distance of a point  $p$  of  $E$  from the line  $D_i$  ( $i=1, 2, \dots, n$ ), then

$$\sum_{i=1}^n \max_{p \in E} d_i^2(p) \leq 3 \max_{p \in E} \left( \sum_{i=1}^n d_i^2(p) \right).$$

If the  $n$  lines are concurrent, the sharper inequality obtained upon replacing the multiplier 3 by 2 is valid. Equality is attained in both cases. The proofs utilize only elementary analytic geometry. L. M. Blumenthal.

Špaček, Antonín. *Note on K. Menger's probabilistic geometry*. Czechoslovak Math. J. 6(81) (1956), 72-74. (Russian summary)

On Menger's paper [Proc. Nat. Acad. Sci. U.S.A. 37

(1951), 226-229; MR 13, 51]; the geometry is a theory of random distance functions in an abstract space. Author gives necessary and sufficient conditions for a random function to be a metric with probability one and shows by an example that Menger's conditions are not sufficient to characterize a random metric in the author's sense.

O. Bottema (Delft).

See also: de Leeuw, p. 294; Loewner, p. 318; Singer, p. 320; Kuiper, p. 329; Cohn, p. 356.

# Differential Geometry

Vyčichlo, F. On pairs of surfaces with common differential invariants. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 6 (1956), 85-97. (Czech. Russian summary)

Let  $(1)x^I(u^I, u^{II})$  and  $(2)x^I(u^I, u^{II})$  be two surfaces referred to the same parameter system  $u^\alpha$  ( $\alpha, \beta = I, II$ ) in the Euclidean three space  $E_3$ . Put  $(a)x_\alpha^I = \partial(a)x^I/\partial u^\alpha$  and

$$(1) \quad \begin{aligned} abn_{k,\alpha\beta} &= abn_{k,[\alpha\beta]} \stackrel{\text{def}}{=} \epsilon_{ijk} (a)x_{[\alpha}^i (b)x_{\beta]}^j, \\ s_{\alpha\beta} &= 2s_{[\alpha\beta]} \stackrel{\text{def}}{=} \delta_{ij} (1)x_{[\alpha}^i (2)x_{\beta]}^j \quad (a, b = 1, 2), \end{aligned}$$

where  $\epsilon$  is the indicator of  $E_3$ . Consider another pair  $(a)y^I(u^I, u^{II})$  of surfaces referred to the parameter system  $u^\alpha$  and denote by  $abn'_{k,\alpha\beta}$ ,  $s'_{\alpha\beta}$  the expressions corresponding to (1). Map the  $y$ 's onto the  $x$ 's by

$$(2) \quad u^\alpha = u^\alpha(u^I, u^{II}).$$

The question arises whether there is such a mapping (2) for which

$$(3) \quad \begin{aligned} abn'_{k,\alpha\beta} \frac{\partial u^\alpha}{\partial u^\alpha} \frac{\partial u^\beta}{\partial u^\beta} &= abn_{k,\alpha\beta}, \\ s'_{\alpha\beta} \frac{\partial u^\alpha}{\partial u^\alpha} \frac{\partial u^\beta}{\partial u^\beta} &= s_{\alpha\beta}. \end{aligned}$$

The author assumes (2) to be known so that

$$(a)y^I(u^I, u^{II}) = (a)x^I(u^I, u^{II}).$$

Then he constructs the expressions  $abn'_{k,\alpha\beta}$ ,  $s'_{\alpha\beta}$  of the  $x$ 's according to (1) and considers instead of (3) the conditions

$$(4) \quad abn'_{k,\alpha\beta} = abn_{k,\alpha\beta}, \quad s'_{\alpha\beta} = s_{\alpha\beta}.$$

A straight forward computation yields necessary and sufficient conditions for (4) (too complicated to be reproduced in a short review). With the exception of some simple cases the form of these conditions depends on the parameters  $u$ 's. {Reviewer's remark: If  $s_{12}, s_{1'2'} \neq 0$  then the radius vectors  $m_k \stackrel{\text{def}}{=} 12n_{k,12}/s_{12}$  and  $m'_k \stackrel{\text{def}}{=} 12n'_{k,1'2'}/s'_{1'2'}$  describe in general two surfaces  $S$  and  $S'$ . If (3) holds then  $S=S'$  and the solution of the problem depends on the possibility (2) of developing  $S$  onto itself.}

V. Hlavatý (Bloomington, Ind.).

Barner, Martin. Geometrische, integralfreie Konstruktion der Asymptotenlinien der Regelflächen. Arch. Math. 7 (1956), 204-213.

Le tangenti alle linee asintotiche di una rigata lungo una sua generatrice appartengono ad una quadrica; esse si rappresentano quindi, sulla quadrica di Klein, in punti di una conica. Quest'ultimo fatto dà modo all'Autore di dare una elegante dimostrazione del problema, già risolto da G. Koenigs della costruzione delle linee asintotiche di una rigata a partire da una di esse. C. Longo (Parma).

Picasso, Ettore. Una proprietà delle linee di Segre e di Darboux. Rend. Sem. Fac. Sci. Univ. Cagliari 26 (1956), 79-82.

See also: Young, p. 316; Pogorelow, p. 330; Soós, p. 330; Segre, p. 334; Takasu, p. 363.

# Riemannian Geometry, Connections

Zalgaller, V. A. On the foundations of the theory of two-dimensional manifolds of bounded curvature. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 575-576. (Russian)

Consider a two-dimensional manifold  $M$  with an intrinsic metric. In a geodesic triangle  $T$  denote as excess  $\epsilon(T)$  the sum of the upper angles minus  $\pi$ . According to the definition of A. D. Alexandrov  $M$  has bounded curvature if for every compact set  $C$  on  $M$  a number  $\beta(C)$  exists such that for any finite set of non-overlapping geodesic triangles  $T_i \subset C$  the inequality  $\sum |\epsilon(T_i)| \leq \beta(C)$  holds. The present note states, with proofs barely indicated, that the following weaker conditions characterize the same metrics: instead of  $C$  take a suitable neighborhood  $U$  of a given point, instead of arbitrary geodesic triangles, convex triangles homeomorphic to circular disks, and very surprisingly, instead of  $\sum |\epsilon(T_i)| \leq \beta(C)$  it suffices to postulate  $\sum \epsilon(T_i) \leq \beta(U)$ , so that there is a restriction only on the triangles with positive excess. H. Busemann.

Budiansky, Bernard; and Pearson, Carl E. A note on the decomposition of stress and strain tensors. Quart. Appl. Math. 14 (1956), 327-328.

The authors prove the following theorems: (1) A symmetrical (stress) tensor  $\sigma_{ij}$  defined in the region  $V$  bounded by the surface  $S$  may be written

$$\sigma_{ij} = \sigma_{ij}' + \sigma_{ij}''$$

where  $\sigma_{ij}'$  and  $\sigma_{ij}''$  have the following properties: (a)  $\sigma_{ij,i}' = -f_j^0$  in  $V$ ,  $\sigma_{ij}' n_i = T_j^0$  on a portion  $S_b$  of  $S$ , where  $f_j^0$  and  $T_j^0$  are prescribed, and  $n_i$  is the unit normal to  $S$ . (b) The (strain) tensor  $\epsilon_{ij}''$  derived from  $\sigma_{ij}''$  by the Hookean relation

$$\epsilon_{ij}'' = L(\sigma_{ij}'') = \frac{1}{E} [(1+\nu)\sigma_{ij}'' + \nu\sigma_{kk}''\delta_{ij}]$$

is related to some (displacement) vector field  $u_i''$  by  $\epsilon_{ij}'' = \frac{1}{2}(u_{i,j}'' + u_{j,i}'')$ , where  $u_i''$  takes on the prescribed value  $u_i^0$  on  $S_A = S - S_b$ . (b) a symmetrical (strain) tensor  $\epsilon_{ij}$  defined in  $V$  may be written  $\epsilon_{ij} = \epsilon_{ij}' + \epsilon_{ij}''$ , where  $\epsilon_{ij}'$  and  $\epsilon_{ij}''$  have the properties: (c)  $\epsilon_{ij}' = \frac{1}{2}(u_{i,j}' + u_{j,i}')$ , where  $u_i'$  is a (displacement) vector that takes on the values  $u_i^0$  on  $S_A$ , (d) the (stress) tensor  $\sigma_{ij}' = H(\epsilon_{ij}')$ , where  $H$  is the inverse of  $L$ , satisfies  $\sigma_{ij,i}' = -f_i^0$  in  $V$ ,  $\sigma_{ij}' n_i = T_j^0$  on  $S^B$ . M. Pini.

Širokov, P. A. On the theory of symmetric spaces. Uč. Zap. Kazan. Univ. 115 (1955), no. 14, 3-19. (Russian)

This paper by the late Professor A. P. Širokov, prepared by his son, deals with Einstein  $V_4$ ,  $R_{\alpha\beta} = R_{\alpha\beta}^0 = k g_{\alpha\beta}$ ,  $k = \text{const}$ , for which  $R_{\alpha\beta\gamma\delta,0} = 0$ , hence  $R_{\alpha\beta\gamma\delta}^0 + R_{\beta\gamma\alpha\delta}^0 + R_{\gamma\alpha\beta\delta}^0 + R_{\alpha\delta\beta\gamma}^0 = 0$ . When  $R_{1414} = p_{11}$ ,  $R_{1424} = p_{11}$ , etc.  $(14 \rightarrow 1, 24 \rightarrow 2, 34 \rightarrow 3, 23 \rightarrow 4, 31 \rightarrow 5, 12 \rightarrow 6)$ ,  $p_{ij} = p_{i+3,j+3}$ ,  $p_{i,j+3} = p_{i+3,j}$ ,  $i, j = 1, 2, 3$ ;  $q_{ij} = p_{i,j+3}$ , then  $p_{ij} = p_{ji}$ ,  $q_{ij} = q_{ji}$ . If  $r_{ij} = p_{ij} + q_{ij}$ ,  $s_{ij} = p_{ij} - q_{ij}$ , then there are 3 types of such spaces: 1) rank of  $r_{ij}$  and  $s_{ij}$  is 3, 2) rank of  $r_{ij}$  is 3, of  $s_{ij}$  is 1, 3) rank of  $r_{ij}$  and  $s_{ij}$  is 1.

The first type is of constant curvature and admits at a point a  $G_3$  of rotations, the second type admits a  $G_4$  the third type a  $G_2$ . A very detailed study of the spaces of the second type follows.

D. J. Struik.

**Kobayashi, Shoshichi. Induced connections and imbedded Riemannian spaces.** Nagoya Math. J. 10 (1956), 15-25.

Let  $O(n)$  be the orthogonal group in  $n$  variables, and let  $M_{n,k} = O(n+k)/O(n) \times O(k)$ ,  $P_{n,k} = O(n+k)/(1) \times O(k)$ , so that  $M_{n,k}$  can be identified with the Grassmann manifold of all  $n$ -planes through the origin of the Euclidean space  $R^{n+k}$  of dimension  $n+k$ , and  $P_{n,k}$  with the Stiefel manifold of all orthonormal  $n$ -frames. Then  $P_{n,k}$  is a principal fiber bundle over  $M_{n,k}$  with the group  $O(n)$  and has a canonical connection. If  $M'$  is an  $n$ -dimensional Riemannian manifold imbedded in  $R^{n+k}$ , there is a mapping  $h: M' \rightarrow M_{n,k}$  defined by assigning to every point  $x$  of  $M'$  the  $n$ -plane through the origin parallel to the tangent plane to  $M'$  at  $x$ . The author gives the proof that the connection on  $M'$  induced by  $j$  is the connection defined by Levi-Civita.

S. Chern (Chicago, Ill.).

**Gu, Čao-Hao. On the problem of imbedding an  $m$ -dimensional set of paths in projective space.** Acta Math. Sinica 5 (1955), 369-381. (Chinese. Russian summary)

The following three theorems are proved: 1) Any  $m$ -dimensional path-space can be regarded as an  $m$ -dimensional space  $V_m$  in a projective space of  $n$  dimensions, with a field of normal spaces  $P_1$  of the first kind, such that the osculating spaces of the paths in  $V_m$  intersect  $P_1$  in a line. 2) Let  $\Pi'_{ik}$  of order  $\nu$ , be the projective connection of a system of paths, and  $\Gamma'_{ik}$  an affine connection of order  $\nu$  with  $\Pi'_{ik}$  as its projective connection. Then the order of the affine connection, which is in projective correspondence with  $\Gamma'_{ik}$  at an inclination  $\phi_i$ , has an affine order  $\nu$  or  $\nu+1$ . Theorem 3 is analogous to 2).

S. Chern (Chicago, Ill.).

**Moffat, John. Generalized Riemann spaces.** Proc. Cambridge Philos. Soc. 52 (1956), 623-625.

Etude des éléments géométriques d'un espace à  $n$  dimensions réelles, sur lequel est défini un tenseur du second ordre symétrique à valeurs complexes  $g_{\mu\nu}$ . Etude de la connexion complexe définie par:

$$g_{\mu\nu}, \sigma = g_{\mu\nu} \Gamma_{\mu\sigma}^{\nu} + g_{\mu\sigma} \Gamma_{\mu\nu}^{\sigma}.$$

Etude des tenseurs de Riemann-Christoffel et Ricci généralisés (parties réelles et imaginaires), et des identités de Ricci. L'auteur indique que cette variété non-riemannienne sera utilisée pour la construction d'une théorie unitaire où les équations du mouvement sont obtenues par la méthode d'Infeld.

Y. Fourès-Bruhat.

★ **Memoirs of the unifying study of the basic problems in engineering sciences by means of geometry. Vol. I.** Kazuo Kondo, Chairman. Gakujutsu Bunken Fukyukai, Tokyo, 1955. xv+590 pp. \$12.50.

This large volume of 600 pages and double columns is the first of a four-volume report of the cooperative research activities of a group of Japanese applied mathematicians, physicists and engineers. The purpose is to introduce into all phases of engineering the methodologies of combinatorial topology, as well as of modern differential geometry (non-holonomic and non-riemannian), by means of the unifying apparatus of tensor analysis.

The articles of the first volume are arranged under four main headings.

Division A deals with "Linear Geometry and Topology of Networks". In electrical networks each branch of a graph is endowed with an "impedance". As a result, tensor algebra and  $n$ -dimensional affine geometry enter the theory of stationary electrical networks. The role of certain ideal or "intuitive" basis vectors in the analysis, and the role of ideal transformers in the synthesis of networks is pointed out.

Division B summarizes the "Differential Geometry of Engineering Dynamical Systems". Starting with the non-Riemannian electrodynamics of rotating electrical machinery it extends the non-Riemannian aspects to the dynamics of moving aircraft. Rheonomic differential geometry is introduced which treats the time variable as one of the generalized coordinates. The affine geometry of paths is generalized to include topics usually ignored in conventional dynamical treatments, namely a) non-linearity by means of contact transformation, b) dissipation terms by stochastic treatment, and c) hysteresis by a non-holonomic point of view. The theory of Finsler spaces enters at many points.

Division C on "Geometry of Deformations and Stresses" extends Saint-Venant's theory of compatibility. The authors emphasize the fact that elasticity materializes a Riemannian space and that the elastic deformations are holonomic transformations. The incompatibility of strains is measured by the non-vanishing components of the Riemann-Christoffel curvature tensor. For the treatment of light structures, extensively used in the Orient, the "Tension-Field" concept is introduced.

Division D on "Non-holonomic Geometry of Plasticity and Yielding" considers plastic deformations as a generalization of the concept of "tearing" from finite physical structures into the realm of infinitesimals. The non-holonomic object  $\Omega_{\alpha\beta,\gamma}$  which appears in the theory of plastic deformations, can be replaced by a torsion tensor  $S_{\alpha\beta,\gamma}$  because of the appearance of both true and quasi-coordinates. Thus the infinitesimal elements into which a Riemannian space has been torn, are interconnected into a non-Riemannian space. The torsion is thereby recognized as dislocation density. (The same infinitesimal interconnection of Riemannian space-elements into a non-Riemannian space also arises in the theory of accelerated electrodynamical and aerodynamical systems treated in Division B.) The concepts developed are applied to the study of imperfections in crystal structure and of microscopically non-uniform materials.

G. Kron.

**Auslander, Louis. Examples of locally affine spaces.** Ann. of Math. (2) 64 (1956), 255-259.

The author constructs countably many non-homeomorphic compact differentiable 3-manifolds, each of which admits a complete affine connection with curvature and torsion zero. Moreover none of these locally affine manifolds are covered by the torus. These examples display the significant difference between locally affine manifolds and locally euclidean (flat Riemannian) manifolds.

L. Markus (Princeton, N.J.).

**Kashiwabara, Shōbin. On the reducibility of an affinely connected manifold.** Tôhoku Math. J. (2) 8 (1956), 13-28.

A manifold  $M$  with symmetric affine connection is said to be  $R$ -reducible if it admits disjoint  $r$ - and  $s$ -dimensional parallel plane fields ( $r+s=\dim M$ ) and if the corresponding maximal integral manifolds  $R(x)$ ,  $S(x)$  through

$x \in M$  give in the neighborhood of  $x$  a local separation of the connection, i.e. the neighborhood is an affine product. The relations between  $R$ ,  $S$  and  $M$  are studied by methods similar to those used by the reviewer in the case of a Riemannian manifold [Proc. London Math. Soc. (3) 3 (1953), 1-19; MR 15, 159] and several analogous results are given. It is also proved that the  $p$ -dimensional homotopy group of every maximal integral manifold is isomorphic with the  $p$ -dimensional homotopy group of  $M$  under the homomorphism induced by the inclusion map. After considering the simply-connected case in some detail, the author finally considers  $R$ -reducible manifolds with second order fundamental group, and shows that there are three distinct classes of such manifolds, in which (a)  $\pi_1(R)=1$ ,  $\pi_1(S)=1$  for all  $R$  and  $S$ ; (b)  $\pi_1(R)=1$ ,  $\pi_1(S)=2$  for all  $R$  and  $S$ ; and (c)  $\pi_1(R)=1$  for all  $R$  and  $\pi_1(S)=1$  or 2, both values being taken. *A. G. Walker.*

**Barthel, Woldemar.** Extremalprobleme in der Finsler-schen Inhaltsgeometrie. Ann. Univ. Sarav. 4 (1955), 171-183 (1956).

In an  $n$ -dimensional symmetric Finsler space with a line element  $L(x, X)$  satisfying the usual conditions let a positive scalar density  $F(x)$  of weight 1 be used to evaluate volume  $(\int F(x) dx^1 \cdots dx^n)$ . For  $n-1$  vectors  $x_1 \cdots x_{n-1}$  spanning a given hypersurface element determined by

$$p_i = \|\delta_i^r x_1^r \cdots x_{n-1}^{r-1}\|, F(x)p_i = X_i,$$

a vector  $X(x, p)$  normal is the hypersurface element. The dual area of a hypersurface element  $(x, x_p)$  is defined as the volume of the parallelepiped in the tangent space at  $x$  spanned by the surface element and the unit normal vector. Then the dual area of a hypersurface  $x(u^1, \dots, u^{n-1})$  is given by  $\int L(x, X) du^1 \cdots du^{n-1}$ , where  $x_p^i = \partial x^i / \partial u^p$ . An invariant differentiation of a tensor field

$$DT^i(x, X) = dT^i + T^k(\gamma_{kh}^i dx^h + L^{-1} a_{kh}^i dX^h)$$

is defined such that  $Dg_{ik}(x, X) = 0$  and  $a_{(ik)h} = 0$ . This determines the  $a_{kh}^i$  uniquely, but not the  $\gamma_{kh}^i$ . The first variation of arc length, the analogue to Weingarten's equation, and the first variation of dual area are expressed by means of this differentiation. It is shown that the  $\gamma_{kh}^i$  are uniquely determined by the requirement that the geodesics become extremals and the geodesic hypersurfaces with respect to the dual metric become minimal surfaces. It is stated, with a proof to appear elsewhere, that in a Minkowski space the spheres solve the isoperimetric problem for dual area. *H. Busemann.*

**Rund, Hanno.** Hypersurfaces of a Finsler space. Canad. J. Math. 8 (1956), 487-503.

In this paper the author seeks to establish a set of formulas concerning hypersurfaces in Finsler space. He deals with such problems as the generalization of normal curvature, the Dupin indicatrix, and the equations of Bianchi, Gauss and Codazzi to Finsler space.

*L. Auslander (Princeton, N.J.).*

**Nasu, Yasuo.** On spaces with constant curvature. Tensor (N.S.) 5 (1956), 164-186.

Using the reviewer's result that a  $G$ -space with locally flat bisectors has as universal covering space either a spherical space of dimension  $>1$ , or a euclidean space or a hyperbolic space [see The Geometry of geodesics, Academic Press, New York, 1955, pp. 310, 331; MR 17, 779], the author proves that spaces which have flat bisectors in

the large are the simply connected spaces of constant curvature. (But this is obvious; it is well-known that the listed spaces are the only simply connected spaces with constant curvature and they have flat bisectors. A space with flat bisectors in the large has the property that any bisector contains with any two points a segment connecting them, and the reviewer showed [l.c. p. 331] that this property leads to the same spaces.) *H. Busemann.*

**Soós, Gy.** Über Gruppen von Automorphismen in affin-zusammenhängenden Räumen von Linielementen. Publ. Math. Debrecen 4 (1956), 294-302.

This paper is concerned with an affinely connected space of line-elements  $X$  endowed with a Finsler metric. The group of automorphisms of  $X$  contains as subgroups the groups of affine and homothetic transformations. The author shows that the group of homothetic transformations is a subgroup of the group of affine transformations. If  $X$  has constant curvature,  $R \neq 0$ , then no proper infinitesimal homothetic transformation exists. If  $X$  has an  $r$ -parameter group of affine transformations, the author proves conditions which are necessary and sufficient for this group to contain a subgroup of homothetic transformations. *C. B. Allendoerfer.*

**Rapcsák, A.** Über das vollständige System von Differentialinvarianten im regulären Cartanschen Raum. Publ. Math. Debrecen 4 (1956), 276-293.

If any differential invariant of a space can be expressed as a function of the elements of a particular set of such invariants, this set is called a "complete" set of differential invariants. The problem considered in this paper is the determination of such a complete set for a regular Cartan space.

The coordinates of an element of a Cartan space are  $(x^1 \cdots x^n, u_1 \cdots u_n)$  and the fundamental structure is given by a function  $L(x, u)$  which is positive homogeneous of the first degree in  $u$ . The author constructs a system of normal coordinates,  $\bar{x}$ , in this space and defines normal tensors in terms of these. Through the use of these tensors he arrives, by considerable manipulation, at a set of tensors which together with their covariant derivatives form a complete set of differential invariants. The definitions of these tensors are, unfortunately, too complicated to include in a brief review.

*C. B. Allendoerfer (Seattle, Wash.).*

See also: Avakumović, p. 315; Auslander, p. 316; Loewner, p. 318; Kobayashi, p. 328; Bergmann, p. 363.

## Complex Manifolds

**Kreyszig, E.**  $\sigma^{n,k}$ -Prozesse und die Abschliessung komplexer affiner Räume. Wiss. Z. Ernst Moritz Arndt-Universität Greifswald. Math.-Nat. Reihe 5 (1955/56), 151-158.

Let  $M$  be a complex manifold (complex dimension  $n$ ) and  $NCM$  be a regular submanifold (complex dimension  $k$ ); the  $\sigma^{n,k}$ -process considered here is an explicit analytic modification  $(M', N')$  of  $(M, N)$  [cf. Kreyszig, same Z. 5 (1954/1955), 457-463; MR 17, 408], in which  $N'$  is fibred over  $N$  by a complex projective space of dimension  $n-k-1$ . The  $\sigma^{k,0}$ -process is the natural extension of Hopf's  $\sigma$ -process [Rend. Mat. e Appl. (5) 10 (1951), 169-182; MR 13, 861]; the  $\sigma^{n,k}$ -process is defined locally by

considering a coordinate neighborhood in which  $N$  is a linear subvariety, and applying the  $\sigma^{n-k,0}$ -process to all the sections of this neighborhood cut by  $(n-k)$ -dimensional hyperplanes normal to  $N$ . A multiple modification  $M'$  of  $M$  is defined by a finite chain of manifolds  $M, M_1, \dots, M_r, M'$ , so that for each successive pair of elements of the chain, one element is a modification of the other. The author shows that all complex multiple projective spaces of the same dimension are multiple modifications of one another, generated by the  $\sigma^{n,k}$ -processes alone.

R. C. Gunning (Princeton, N.J.).

See also: Švec, p. 306; Grothendieck, p. 327.

### Algebraic Geometry

**Brauner, H.** Konstruktive Durchführung der durch die Sehnen einer Raumkurve 3. Ordnung vermittelten Abbildung des Raumes auf eine Ebene. *Monatsh. Math.* 60 (1956), 231–248.

The authors has previously studied a birational representation of the chords of a twisted cubic curve by the points of a plane [*Monatsh. Math.* 59 (1955), 258–273; MR 17, 895]. Here he completes that study from the constructive point of view, with reference to a convenient metrically specialized cubic; moreover, he makes a number of applications, giving e.g. the representation of a linear complex of lines by means of the chords of such a cubic belonging to the complex. B. Segre (Rome).

**Gallarati, Dionisio.** Sulle ipersuperficie cubiche circoscritte ad una quadrica. *Atti Accad. Ligure* 11 (1954), 161–184 (1955).

The author classifies all possible configurations of a quadric and a cubic hypersurface, both irreducible, in  $S_r$ , and having simple contact at each of their general points of intersection. There are fourteen basic cases, one in  $S_5$ , five in  $S_4$ , seven in  $S_3$ , and one in  $S_2$ ; all others are projections of these. There is one case of double contact.

R. J. Walker (Ithaca, N.Y.).

**Segre, Beniamino.** Alcune osservazioni sulle superficie cubiche nel campo razionale. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 20 (1956), 147–149.

T. Skolem [*Math. Z.* 63 (1955), 295–312; MR 17, 464] ha dato un esempio di superficie cubica razionale (e cioè, a coefficienti razionali), priva di punti multipli, che mediante un sistema omaloideico razionale di quadriche vien trasformata in una superficie cubica avente una terna razionale di punti doppi (irrazionali); ciò consente di ottenere per la data superficie una rappresentazione parametrica razionale. Alla questione, posta da T. Skolem, di vedere se un simile procedimento possa essere esteso alle superficie cubiche generali, B. Segre risponde nella Nota presente, dimostrando che: le superficie cubiche razionali alle quali può estendersi il procedimento suddetto son tutte e sole quelle dotate di tre rette  $a, b, c$  a due a due sghembe, costituenti una terna razionale e dotate di (almeno) una comune retta trasversale razionale (ciascuna delle due schiere rigate della quadrica definita da  $a, b, c$  deve cioè avere infinite generatrici razionali). In particolare la superficie cubica  $\sum a_i x_i^3 = 0$  ( $i=1, 2, 3, 4$ ;  $a_i$  numeri razionali non nulli) può essere trasformata cremonianamente nel campo razionale in una superficie cubica dotata di tre punti doppi non allineati se il pro-

dotto di due opportune delle  $a_i$  diviso per il prodotto delle rimanenti è il cubo di un numero razionale.

D. Gallarati (Genova).

★ **Châtelet, F.** Points rationnels sur les surfaces cubiques. *Séminaire A. Châtelet et P. Dubreil de la Faculté des Sciences de Paris, 1953/1954. Algèbre et théorie des nombres.* 2e tirage multigraphié, pp. 8-01–8-11. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956.

The present paper begins by summarizing some of the arithmetical results on cubic surfaces expounded by B. Segre [*Bull. Amer. Math. Soc.* 51 (1945), 152–161; *Math. Notae* 11 (1951), 1–68; MR 6, 185; 13, 678]. One of them asserts that a rational non-singular cubic surface containing a rational sextuplet of lines can be birationally represented on a plane in the rational field if, and only if, it contains one rational point. The author now remarks that, consequently, the questions of deciding whether the surface does or does not contain such a point, and of obtaining all the rational points when they exist, can be dealt with by using the methods of his thesis [*Ann. Sci. Ecole Norm. Sup.* (3) 61 (1944), 249–300; MR 7, 323]; then he applies this remark to the research of the rational solutions  $(x, y, z)$  of the equation

$$N(x + \theta y + \theta^2 z) = A,$$

where  $A$  is a rational number,  $\theta$  is a cyclic cubic number, and  $N(a)$  denotes the norm of  $a$ ,  $a$  being a number of the field  $R(\theta)$ . Next the rational non-singular cubic surfaces containing a rational triplet are likewise investigated, using the fact that on the surface there are two sextuplets containing the given triplet; this is then applied to the study of the above equation in the case when  $\theta$  is a non-cyclic cubic number. The same purpose is also achieved by another method, in which the theory of ideals of algebraic numbers directly intervenes. B. Segre (Rome).

**Segre, Beniamino.** Invarianti topologico-differenziali, varietà di Veronese e moduli di forme algebriche. *Ann. Mat. Pura Appl.* (4) 41 (1956), 113–138.

A simple point  $P$  of a differentiable variety  $V_d$  is said to be  $n$ -regular if the  $n$ -osculating space to  $V_d$  at  $P$  has dimension  $n + \alpha C_d - 1$ . The author commences with some general remarks concerning the use of this concept in the study of differentiable varieties; and notes that, on a  $V_d$  of differentiable class  $C^n$ , a general set of  $d+1$  linear curve elements  $E_{h+1}$  containing the same  $E_h$  ( $1 \leq h \leq n-1$ ) possess precisely  $d-1$  'topological' invariants.

By means of the Veronese transformation  $X_{i_1, \dots, i_d} = x_0^{i_1} x_1^{i_2} \dots x_d^{i_d}$  ( $i_0 + i_1 + \dots + i_d = n$ ) a primal of degree  $n$  in  $S_d$  can be mapped into a prime section of the Veronese variety  $V_d^{(n)}$  in  $S_r$  ( $r = n + \alpha C_d - 1$ ), which has the property of being  $n$ -regular at every point. A module of forms of degree  $n$  is mapped into a space of  $S_r$ . In the case of modules generated by  $h$  forms  $K_1, K_2, \dots, K_h$  of degrees  $k_1, k_2, \dots, k_h$  ( $1 \leq h \leq d+1$ ), such that the primals  $K_i = 0$  intersect (regularly) in a variety of dimension  $d-h$ , the author shows that the dimension of the corresponding space of  $S_r$  can be easily calculated. In particular when  $h=d$  and  $n \geq k_1 + k_2 + \dots + k_d - d$  the corresponding space  $S_{p-1}$  has dimension  $p-1 = \prod_{i=1}^d k_i - 1$ ; moreover if  $P_1, P_2, \dots, P_s$  are the isolated intersections, with multiplicities  $p_1, p_2, \dots, p_s$ , of the  $d$  primals, then, for  $n \geq p-1$ ,  $S_{p-1}$  is shown to be the join of spaces  $S_{p_i-1}$  passing through  $O_i$  (the transform of  $P_i$  on  $V_d^{(n)}$ ) and lying in the  $(p_i-1)$ -osculating space to  $V_d^{(n)}$  at  $O_i$  ( $i=1, 2, \dots, s$ );  $S_{p-1}$  is in

fact the intersection of  $S_{p-1}$  with this osculating space. When the tangent primals to  $K_1, K_2, \dots, K_d$  at  $P_i$  have no generator in common (i.e. in the simple case),  $K_i$  has multiplicity  $t_j$  at  $P_i$  ( $j=1, 2, \dots, d$ ) and  $n \geq t_1 t_2 \dots t_d + t_1 + t_2 + \dots + t_{d-1} - d$ , then  $S_{p-1}$  lies in the  $(t_1 + t_2 + \dots + t_d - d)$ -osculating space to  $V_d^{(n)}$  at  $O_i$ .

From these geometric properties the author deduces that, if a primal  $K=0$ , of degree  $n$ , in  $S_d$  has multiplicity  $p_i$  at  $P_i$ , then  $K$  can be written in the form  $\sum A_i K_i$ , where  $A_i$  is of degree  $n - k_i$ ; further in the case of the points  $P_i$  at which the primals present the simple cases it is sufficient for  $K$  to have the corresponding value of  $t_1 + t_2 + \dots + t_d - d + 1$  as multiplicity at  $P_i$ . Similar results, for  $h$  primals ( $1 \leq h < d$ ) which intersect regularly are proved by reduction to the zero dimensional case.

D. Kirby (London).

**Lluis, Emilio.** On the singularities which appear in projecting algebraic varieties. Bol. Soc. Mat. Mexicana (2) 1 (1956), 1-9. (Spanish)

Siano  $V$  una varietà algebrica non singolare appartenente allo spazio proiettivo ad  $n$  dimensioni  $S_n$ , definita sopra un corpo  $k$  di caratteristica  $p$ ,  $V'$  una generica proiezione di  $V$  sopra uno spazio lineare  $S_m$  in  $CS_m$ . L'Autore studia le singolarità di  $V'$  e si occupa in particolare della trasformazione birazionale di  $V$  in un'ipersuperficie dotata di singolarità di tipo semplice, mettendo in luce come certe proprietà dipendano dalla caratteristica di  $k$ . Dopo alcuni risultati preliminari l'Autore dimostra che: a) se il punto  $Q'$  di  $V'$  è proiezione di  $\mu$  punti semplici  $Q_1, Q_2, \dots, Q_\mu$  di  $V$ ,  $Q'$  è origine di  $\mu$  falde di  $V'$ , essendo falde lineari quelle che provengono da punti  $Q$  tali che lo spazio in essi tangente a  $V$  non incontra il centro di proiezione; b) se  $V$  ha dimensione  $r$  e non appartiene a spazi lineari di dimensione  $< n$ , esiste un suo modello birazionale  $V^{(0)} \subset CS_m$ , ottenuto per proiezione, su cui vi è una catena discendente di varietà algebriche  $V^{(1)} \supset V^{(2)} \supset \dots \supset V^{(n-m)}$ ,  $V^{(i)}$  di dimensione  $\leq r - i(m-r)$ , tali che i punti di  $V^{(i-1)} - V^{(i)}$  sono origini di  $i$  falde, le quali sono lineari (eccetto per sottovarietà). Inoltre se  $p=0$  i punti di  $V^{(n-m)}$ , eccetto sottovarietà, sono origini di  $n-m+1$  falde. L'A. esamina in particolare il caso in cui esista un modello birazionale di  $V$  privo di punti singolari ed appartenente ad  $S_{2r+1}$ . Ad esempio, ritrova che ogni curva algebrica, definita sopra un campo di caratteristica zero ammette un modello piano con un numero finito di nodi; non così se  $p>0$ , come appare da un esempio di

curva algebrica di  $S_3$  le cui corde son tutte  $p$ -secanti. Per le superficie si ritrova, se  $p=0$ , l'esistenza di un modello di  $S_4$  con un numero finito di punti doppi e d'un modello di  $S_5$  dotato di curva doppia con un numero finito di punti tripli.

D. Gallarati (Genova).

**Spampinato, Nicolò.** Prolungamento di una falda dell'  $S_3$  complesso nell'algebra dei numeri triduali. Giorn. Mat. Battaglini (5) 4(84) (1956), 49-67.

**Godeaux, Lucien.** Sur les points de diramation triples d'une surface multiple. Rev. Un. Mat. Argentina 17 (1955), 39-52 (1956).

**Gallarati, Dionisio.** Restituzione di priorità. Boll. Un. Mat. Ital. (3) 11 (1956), 382-383.

A statement by the author that some of the results in his article "Alcune osservazioni sopra le varietà i cui spazi tangenziali si appoggiano irregolarmente a spazi assegnati" in Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Mat. (8) 20(1956), 193-199 had already been found by A. Terracini in "Superficie particolari dello spazio a cinque dimensioni in relazione con le loro linee principali" in Annali di Matem., (4) 17 (1938), 23-44.

**Turri, Tullio.** Trasformazioni involutorie con stella unita di rette in un  $S_r$ . Rend. Sem. Fac. Sci. Univ. Cagliari 26 (1956), 58-63.

**Turri, Tullio.** Sulla determinazione dei gruppi di moltiplicabilità delle matrici di Riemann. Rend. Sem. Fac. Sci. Univ. Cagliari 26 (1956), 64-67.

**Balsimelli, Pio.** Trasformazioni birazionali legate alle proiettività dell' $S_1$  tricompleso e triduale. Giorn. Mat. Battaglini (5) 4(84) (1956), 75-80.

**Turri, Tullio.** Sulle trasformazioni birazionali cicliche dello spazio a periodo pari. Rend. Sem. Fac. Sci. Univ. Cagliari 25 (1955), 119-129 (1956).

**Turri, Tullio.** Inesistenza di trasformazioni birazionali involutorie determinanti un complesso tetraedrale. Rend. Sem. Fac. Sci. Univ. Cagliari 25 (1955), 130-133 (1956).

See also: Leicht, p. 285.

## NUMERICAL ANALYSIS

### Numerical Methods

**Flemming, D. P.** Iterative procedure for evaluating a transient response through its power series. Math. Tables Aids Comput. 10 (1956), 73-81.

The author presents an iterative method of determining the MacLaurin expansion of a function whose Laplace transform is a given rational function. Thus given that

$$\int_0^\infty e^{-st} F(t) dt = \frac{A_0 S^{N-1} + A_1 S^{N-2} + \dots + A_{N-1}}{S^N + b_1 S^{N-1} + \dots + b_N}$$

we wish to determine  $\alpha_0, \alpha_1, \dots$  such that

$$(1) \quad F(t) = \sum_{j=0}^\infty \alpha_j t^j / j!$$

The  $\alpha$ 's are found recursively via the equations

$$(2) \quad \begin{aligned} A_k^{(0)} &= A_k, \quad \alpha_0 = A_0^{(0)}, \\ A_k^{(j+1)} &= A_{k+1}^{(j)} - \alpha_j b_{k+1}, \\ b_{k+1} &= A_0^{(k+1)}. \end{aligned}$$

In practice one should know how many  $\alpha$ 's to compute to obtain a prescribed degree of accuracy over a given range of  $t$ . If it is required that the error committed in truncating the series (1) should not exceed  $\epsilon$  in absolute value for all  $t$  between 0 and  $t_0$ , then it suffices to take more than

$$e t_0 \beta + \log A \beta (\beta^N - 1) - \log(2(\beta - 1)(\epsilon - 1)\epsilon)$$

terms of (1). Here  $\beta - 1 = \max |b_j|$ ,  $A = \max |A_j|$ . These results follow from the inversion theorem and the fact

that the transform of  $\theta/j!$  is  $S^{-1-j}$ . There is no discussion of rounding errors or possible loss of significance in applying the iterative formulas (2). *D. H. Lehmer.*

**Wall, D. D. Note on predictor-corrector formulas.** *Math. Tables Aids Comput.* 10 (1956), 167.

The author recommends that only one application of the corrector formula should be used with a predictor-corrector method for solving an ordinary differential equation. An example is given to show that repeated use of a corrector formula may actually yield poorer results than a single use. The author feels that if the difference between the predicted value and the first corrected value is too large, then instead of repeating the correction process one should reduce the interval size.

{It is not indicated for which predictor-corrector methods the above considerations apply. It should be noted that if the order of accuracy of the predictor formula is less than that of the corrector formula, then more than one use of the latter may indeed be advisable.}

*D. M. Young, Jr. (Los Angeles, Calif.).*

**Wall, D. D. The order of an iteration formula.** *Math. Tables Aids Comput.* 10 (1956), 167-168.

In this note it is shown that the method of false position (repeated linear inverse interpolation) for solving an equation  $f(x)=0$  is of order  $\frac{1}{2}(1+\sqrt{5})=1.6+$ . The close relation of this method to Newton's method, which is of order 2, is shown. There is a line omitted in the first paragraph. *D. M. Young, Jr. (Los Angeles, Calif.).*

**Flemming, D. P. An iterative method for Taylor expansion of rational functions, and applications.** *Math. Tables Aids Comput.* 10 (1956), 120-130.

A simple iterative procedure is set up for calculating the coefficients of the Taylor expansion of a rational function about any point in the complex plane other than the zeros of the denominator. Examples are given showing its use in Laplace transforms with simple and multiple poles to get the partial fraction expansion, and to get the power series solution, as well as getting a "distribution function" from a rational "generating function". The paper includes a discussion of the truncation errors due to cutting off the Taylor expansion after a finite number of terms and a very brief mention of roundoff errors. *R. W. Hamming.*

**Dupač, Václav. Stochastic numerical methods.** *Časopis Pěst. Mat.* 81 (1956), 55-68. (Czech. Russian and English summaries)

It is partly an expository paper on Monte Carlo methods in numerical calculation. Following topics are treated: Integration, evaluation of volumes, matrix inversion, determination of  $\pi$ . Original results: Let  $M$  be a  $r$ -dimensional body, bounded by a rectifiable surface, and contained in a cube, which is divided into  $n$  smaller cubes; let a point be chosen at random in each of these cubes; then the ratio of points lying in  $M$  is an unbiased estimate of the volume of  $M$  with  $\sigma = O(n^{-(r+1)/2r})$ . In discussion on Forsythe-Leibler matrix inversion method, upper bounds containing only known quantities are given for variances of estimators and for average length of realisations. A modified version of Buffon's needle problem is discussed in the last section. *J. Janko (Praha).*

**Hammersley, J. M.; and Morton, K. W. A new Monte Carlo technique: antithetic variates.** *Proc. Cambridge Philos. Soc.* 52 (1956), 449-475.

In a previous paper [J. Roy. Statist. Soc. Ser. B.

16 (1954), 23-38; MR 16, 287] the authors describe some Monte Carlo calculations for which special techniques yield estimates with variances considerably below those obtainable by a straightforward, simple-minded approach. In the next paper is described in rather general terms a particular technique for obtaining such a reduction. Here this special technique, "antithetic variates", is illustrated and compared with a somewhat analogous but less powerful method of "controlled variates".

The application is to integration in Euclidean space. Suppose a parameter  $\theta$  is to be estimated and  $t$  is a Monte Carlo estimator. In the method of controlled variates, one selects a statistic  $t'$  whose expectation  $E(t')$  is known, and which is positively correlated with  $t$ , and uses  $t - t' + E(t')$  as estimator. In the method of antithetic variates one selects a statistic  $t''$  whose expectation is also  $\theta$ , but which is negatively correlated with  $t$ , and uses  $\frac{1}{2}(t + t'')$  as estimator. More generally, if  $E(t_i) = \theta$ , one could use a weighted mean of the  $t$ 's as estimator.

Techniques for selecting estimators are discussed, and the estimation of a six-dimensional integral is worked out in some detail. Since the application of the method requires a considerable amount of analysis and ingenuity it is hard to judge its utility in general, but for the cases considered a vast reduction in the variance is achieved, and the paper represents a major contribution to the study of Monte Carlo Methods.

*A. S. Householder.*

**Franckx, Ed. La méthode de Monte-Carlo.** *Assoc. Actuaire Belges. Bull. no. 58* (1956), 89-101.

An expository account of simple application of the Monte Carlo method.

*John Todd (Washington, D.C.).*

**Michalup, Erich. Some approximation formulae of the effective rate and the force of interest.** *Skand. Aktuarietidskr.* 38 (1955), 163-164.

Simple approximation formulae are given for the conversion of  $i$  to  $\delta$  and vice versa ( $\delta = \ln(1+i)$ ).

$$(A) \quad \delta = 2i[15(2+i)^2 - 4i^2]/(2+i)[15(2+i)^2 - 9i^2];$$

$$(B) \quad \delta =$$

$$2i(2+i) \frac{1155(2+i)^4 - 1190i^2(2+i)^2 + 231i^4}{1155(2+i)^6 - 1575i^2(2+i)^4 + 525i^4(2+i)^2 - 25i^6};$$

$$(C) \quad i = 2\delta(60 + \delta^2)/[12(10 + \delta^2) - \delta(60 + \delta^2)];$$

$$(D) \quad i = \frac{2\delta[420(36 + \delta^2) + \delta^4]}{30[112(9 + \delta^2) + \delta^4] - \delta[420(36 + \delta^2) + \delta^4]}.$$

The error does not exceed  $3.10^{-11}$ , (A);  $5.10^{-21}$ , (B);  $2.10^{-12}$ , (C);  $2.10^{-21}$ , (D); at the rate of 10%

(A) and (B) are the second and the fifth approximants of

$$\delta = 2 \times \left[ \frac{1}{1} - \frac{x^2}{3} - \frac{4x^2}{5} - \frac{9x^2}{7} - \dots \right] \quad \left( x = \frac{i}{2+i} \right).$$

(C) and (D) are the second and the fourth approximants of Lambert's continued fraction

$$x = \tanh y = \frac{y}{1} + \frac{y^2}{3} + \frac{y^2}{5} + \dots \quad \left( y = \frac{\delta}{2} \right).$$

*S. C. van Veen (Delft).*



Introduction to the theory of divergent series, Hafner, New York, 1948; MR 6, 45; 10, 31] is a satisfactory method for summing the oscillating series that have to be evaluated. R. P. Boas, Jr. (Evanston, Ill.).

**Goodwin, E. T.** Note on the computation of certain highly oscillatory integrals. Math. Tables Aids Comput. 10 (1956), 96-97.

The author suggests calculating integrals of the form  $\int_0^\infty f(t)g(t) \cos ut \, dt$  by calculating instead the equal integral  $\frac{1}{2} \int F(y)\{G(u+y)+G(|u-y|)\}dy$ , where  $F$  and  $G$  are the cosine transforms of  $f$  and  $g$ . As an example he gives  $f=(t^2-1)^{-1}$ ,  $g=e^{-at}$ . R. P. Boas, Jr. (Evanston, Ill.).

**Smith, Ed S.** Men vs. machines on quadrature in weapons analysis. Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Md. vol. 3 (1956), no. 3, pp. 6-14. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D.C.)

**Keitel, Glenn H.** An extension of Milne's three-point method. J. Assoc. Comput. Mach. 3 (1956), 212-222.

The familiar Milne's method for solving ordinary differential equations, which is based on the use of a predictor formula and Simpson's integration formula as a corrector formula, is modified to permit halving or doubling of the integration interval. This method can be used on a high speed computer and results in the use of the largest interval size consistent with the required accuracy, thus saving a considerable amount of computation time. Special formulas are developed to permit halving or doubling the integration interval without requiring additional computer storage. This saving, however, will become of less importance as the storage capacities of modern computers increase. At each step in the integration procedure the decision is made automatically whether to double, halve, or keep the same integration interval and depends on the difference between the results obtained by using the predictor formula and by the first application of the corrector formula. Applications of the method to two problems arising in the study of the scattering of radio waves are described. It is estimated that from 16 to 256 times as much computing time would have been required using Milne's method with a fixed step size. D. M. Young, Jr. (Los Angeles, Calif.).

**Dahlquist, Germund.** Convergence and stability in the numerical integration of ordinary differential equations. Math. Scand. 4 (1956), 33-53.

Let

$$\rho(\zeta) = \sum_{h=0}^k \alpha_h \zeta^h, \quad \sigma(\zeta) = \sum_{h=0}^k \beta_h \zeta^h, \quad Ey(x) = y(x+h),$$

$$Ly(x) = \rho(E)y(x) - h\sigma(E)y'(x),$$

where  $h$  is a parameter. The order of  $L$  is  $k$  and the degree  $p^*$  of  $L$  is the largest value of  $p$  for which  $Ly(x) = O(h^{p+1})$ . An operator  $L$  is stable if  $p^* \geq 1$ ,  $\rho(\zeta_0) = 0$  implies  $|\zeta_0| \leq 1$ , and  $\rho(\zeta_0) = \rho(\zeta_1) = 0$ ,  $|\zeta_0| = |\zeta_1| = 1$  implies  $\zeta_0 \neq \zeta_1$ . Theorem. If  $L$  is a stable operator, then  $p^* \leq k+2$  and  $p = k+2$  if and only if the polynomial

$$R(z) = [\frac{1}{2}(z-1)]^k \rho(\zeta), \quad \zeta = (z+1)(z-1)^{-1},$$

is an odd function and  $R(z) = 0$  only on the imaginary axis. If  $k$  is odd, then  $p^* \leq k+1$ . Consider the differential equation (1)  $y' = f(x, y)$ ,  $y(a) = y_0$ ,  $a \leq x \leq b$ , where

$$y(x) \in C^{(p^*+1)}[a \leq x \leq b]$$

and

$$f(x, y) \in C^2[a \leq x \leq b, |y - y(x)| \leq r],$$

and the corresponding difference equation

$$(2) \quad \rho(E)y_n - h\sigma(E)f_n = 0,$$

where  $x_n = a + nh$  and  $f_n = f(x_n, y_n)$ . Consider the class  $P'(h, \alpha, \beta)$  of perturbed equations

$$(3) \quad \rho(E)\tilde{y}_n - h\sigma(E)f_n = \eta_n,$$

where  $\tilde{y}_j = y_j + \theta_j$  ( $j=0, 1, \dots, k-1$ ),  $f_n = f(x_n, \tilde{y}_n)$ , and  $|\eta_n| < \alpha h^\beta$ ,  $|\theta_j| < h^\beta$ . The numbers  $\eta_n$  and  $\theta_j$  correspond to round-off errors in calculating  $y_n$  and errors in the initial values  $y_j$ , respectively.  $\tilde{y}_n$  converges stably ( $P'$ ) to the solution of (1) if

$$\limsup_{h \rightarrow 0} \sup_{P'(h, K, \beta)} \sup_{a \leq x \leq b} |\tilde{y}_n - y(x)| \leq K\varepsilon,$$

$K$  independent of  $\varepsilon$ . Theorem.  $\tilde{y}_n$  converges stably to the solution of (1) if and only if  $L$  is stable. Other theorems concerning stable operators  $L$  are proved. J. Hale.

**Lotkin, Mark.** A note on the midpoint method of integration. J. Assoc. Comput. Mach. 3 (1956), 208-211.

L'auteur propose d'intégrer l'équation

$$y' = f(x, y)$$

par la méthode suivante

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}(y_n - y_{n-1}), \quad y_{n+1} = y_n + h y_{n+\frac{1}{2}}.$$

Cette méthode qui ne requiert qu'un calcul de  $f(x, y)$  par pas est d'ordre 3. Cette méthode est comparée sur un exemple simple avec la méthode d'Euler-Cauchy qui est du même ordre, mais requiert 2 calculs de  $f(x, y)$ .

J. Kuntzmann (Grenoble).

**Todd, John.** A direct approach to the problem of stability in the numerical solution of partial differential equations. Comm. Pure Appl. Math. 9 (1956), 597-612.

Consider the equation (1)  $u_{xx} = u_t$ ,  $0 < x < 1$ ,  $t \geq 0$ ,  $u(0, t) = 0$ ,  $u(1, t) = 0$ ,  $t \geq 0$ ,  $u(x, 0) = f(x)$ . For a given difference equation analogue of (1), the problem of interest is to determine the effect of a line of errors on the later computed values in the difference equation. If the errors in the later computed values are uniformly bounded, then the difference equation is stable. Expressing the difference equation for a given discretization in matrix form and expressing the errors in terms of the eigenvalues and eigenvectors of the corresponding matrices, the author discusses the problem of stability for the above diffusion equation for one and two dimensions, the wave equation and the equation of the vibrating bar. J. K. Hale.

See also: Choquet, p. 288; Wynn, p. 301; Goldberg and Varga, p. 304; Sobolev, p. 322; Salzer, p. 339; Czaykowski, p. 352; Hay and Eggington, p. 352; Jones, p. 356; Sedney, p. 356; Bracewell, p. 365.

### Graphical Methods, Nomography

See: Hasse, p. 311; Royston, p. 346; Kornecki, p. 350; Hay and Eggington, p. 352.

Tables

**Salzer, Herbert E.** Coefficients for complex osculatory interpolation over a Cartesian grid. *J. Math. Phys.* 35 (1956), 152-163.

Le présent article donne les valeurs exactes des coefficients d'interpolation de Lagrange-Hermite dans le cas où la fonction et sa dérivée 1ère sont connues.

*J. Kuntzmann (Grenoble).*

**Horgan, R. B.** Radix tables for  $\sin x$  and  $\cos x$ ,  $x = a.10^k$  degrees,  $a = 1(1)9$ ,  $k = -3(1)1$ . *Math. Tables Aids Comput.* 10 (1956), 164-166.

The table gives  $\sin x$  and  $\cos x$  for  $x$  in degrees ranging as follows

$$x = .001(.001).01(.01).1(.1)1(1)10(10)90$$

to twenty decimals. They are intended to be used in connection with the formulas

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta,$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta,$$

and their generalizations to arguments which are sums of up to 5 terms, all duly written out. The table was calculated ab initio from power series by the SWAC. To obtain a sine or cosine of an angle like 12.345 degrees by the method described requires 64 multiplications and 15 additions. If an iterative method is used only 14 multiplications and 7 additions are required. *D. H. Lehmer.*

**Ashour, A.; and Sabri, A.** Tabulation of the function

$$\psi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}.$$

*Math. Tables Aids Comput.* 10 (1956), 57-65.

The function of Spence,  $\psi(\theta)$  mentioned in the title and given also by the integral

$$\psi(\theta) = - \int_0^{\theta} \log(2 \sin(t/2)) dt$$

is tabulated to six decimals for  $\theta = 0^\circ(10')180^\circ$  together with first differences. A previous table of Clausen [*J. Reine Angew. Math.* 8 (1832), 298-300] had been found inadequate for application purposes. The present table is pretty nearly linear for  $\theta$  in the second quadrant. Near the origin interpolation is difficult. *D. H. Lehmer.*

**Tsao, Chia Kuei.** Distribution of the sum in random samples from a discrete population. *Ann. Math. Statist.* 27 (1956), 703-712.

Tables are given for the distribution of the sums,  $S = \sum_{i=1}^m X_i$ , where the  $X_i$ 's denote a random sample

drawn from a multinomial population. The tables cover certain selected cases where there are  $k$  classes,  $k = 3(1)6$ , and the probability associated with each class is  $1/k$ .

The author notes that a referee has pointed out that portions of his tables have been previously calculated by J. W. Whitfield [*British J. Statist. Psychol.* 6 (1953), 35-40]. The author also notes that several entries are in disagreement. *M. Muller (Princeton, N.J.).*

**David, H. A.** Revised upper percentage points of the extreme studentized deviate from the sample mean. *Biometrika* 43 (1956), 449-451.

The tables by K. R. Nair [*Biometrika* 35 (1948), 118-144; 39 (1952), 189-191; *Biometrika* tables for statisticians, vol. I, Cambridge, 1954, Table 26; MR 9, 602; 13, 961; 16, 53] are corrected by using a better approximation.

*W. Hoeffding (Chapel Hill, N.C.).*

See also: Tatarkiewicz, p. 284; Selmer, p. 285; Allen, p. 300; Krarup and Svejgaard, p. 337; Broadbent, p. 340; Noether, p. 345; Coles, p. 355; Steel, p. 356.

Machines and Modelling

**Lebedev, S. A.** The high-speed electronic calculating machine of the Academy of Sciences of the U.S.S.R. *J. Assoc. Comput. Mach.* 3 (1956), 129-133.

This is a description of BESM, which is a three-address parallel floating binary machine, with a high-speed cathode ray tube memory of 1023 words, supplemented by a magnetic drum containing 5120 words, and magnetic tapes. There is also a special fixed germanium-diode storage of 276 words which is apparently used to contain sub-routines. A word consists of 32 bits: a sign, 5 bits for the mantissa and 26 bits for the characteristic. The machine is about a 0.1 millisecond one (including access times to the various storages). There are 31 instructions. Input is by photoelectrically read tape, or magnetic tape; output is on a typewriter (15 words/sec.) or on magnetic tape from which visible records can be made by photographic methods (200 words/sec.). *John Todd.*

**Truter, Mary R.** The use of a "506" Hollerith (Bull) multiplying punch for crystallographic calculations. *Proc. Leeds Philos. Lit. Soc. Sci. Sect.* 6 (1954), 140-153.

See also: Šestakov, p. 272; Keitel, p. 338; Horgan, p. 339; Eringen, p. 350; Steel, p. 356; Cohn, p. 356; Foster and Rapoport, p. 366.

PROBABILITY

**Rényi, Alfréd.** On a new axiomatic theory of probability. *Acta Math. Acad. Sci. Hungar.* 6 (1955), 285-335. (Russian summary)

It is proposed to define conditional probability as the primary concept. Let  $S$  be an arbitrary set,  $\mathfrak{A}$  a  $\sigma$ -algebra of subsets of  $S$ ,  $\mathfrak{B}$  a non-empty subset of  $\mathfrak{A}$ . A function  $P(A|B)$  of two set variables is defined for  $A \in \mathfrak{A}$  and  $B \in \mathfrak{B}$  satisfying the axioms (I)  $P(A|B) \geq 0$ ,  $P(B|B) = 1$ ; (II) for each  $B$ ,  $P(\cdot|B)$  is a measure;

$$(III) \quad P(A|BC)P(B|C) = P(AB|C)$$

if  $A \in \mathfrak{A}$ ,  $B, C$  and  $BC \in \mathfrak{B}$ . If so the collection  $[S, \mathfrak{A}, \mathfrak{B},$

$P(\cdot|\cdot)]$  is called a conditional probability space. Various simple consequences of the axioms and alternative forms of (III) are given. A sufficient condition is given so that  $P(A|B) = Q(AB)/Q(B)$ , where  $Q$  is a measure on  $\mathfrak{A}$  and  $Q(B) > 0$  (cf. the next review). Special cases of conditional probability spaces are discussed; for example in spaces with ordered dimensions like Euclidean spaces  $P(A|B)$  may reduce to the quotient form based on a measure in the proper dimension of  $B$ . This includes the "Cavalieri spaces" in solid geometry where the conditioning is the sectioning by hyperplanes and the Cavalieri principle of comparing volumes is satisfied. Many illustrations are

taken from "ordinary probability theory". The typical situation is when the conditional probability can be expressed as the limit of a quotient leading to indeterminate forms of the type  $0/0$  or  $\infty/\infty$ . A typical example is the result of Erdős and the reviewer that if  $S_n$  is the sum of  $n$  independent, identically distributed, integer-valued random variables of span one and mean zero, then

$$P(S_n=a)/P(S_n=b)$$

tends to one as  $n \rightarrow \infty$  for any two integers  $a$  and  $ab$ . In the present terminology  $S_n$  tends to the uniform conditional probability distribution on the set of all integers. Similar phenomena have been discovered for Markov chains. A conditional law of large numbers is proved leading to an extension of Borel's normal numbers to Cantor's series. (Despite the interest of the paper it remains to be seen if these notions furnish more than a convenient viewpoint. The existence of the limit in (52) corresponding to an extension of the aforementioned result to a general Markov chain, has been negated by an example due to F. J. Dyson and the reviewer (unpublished).)

K. L. Chung (Chicago, Ill.).

**Császár, Ákos.** Sur la structure des espaces de probabilité conditionnelle. Acta Math. Acad. Sci. Hungar. 6 (1955), 337-361. (Russian summary)

The notations are the same as in the preceding review. A family  $\{\mu_\alpha\}$  of measures generate the conditional probability space  $[S, \mathfrak{A}, \mathfrak{B}, P(\cdot|\cdot)]$  in case each measure  $\mu_\alpha$  is defined on  $\mathfrak{A}$ ; to each  $B \in \mathfrak{B}$  corresponds at least one index  $\alpha$  such that  $0 < \mu_\alpha(B) < \infty$  and for each such  $\alpha$  we have  $P(A|B) = \mu_\alpha(AB)/\mu_\alpha(B)$  for every  $A \in \mathfrak{A}$ . Such a family is dimension-ordered in case the indices  $\{\alpha\}$  form an ordered set and  $\mu_\alpha(A) < \infty$  and  $\alpha < \beta$  implies  $\mu_\beta(A) = 0$ . Necessary and sufficient conditions are given for a conditional probability space to be generated by a family of measures; by a dimension-ordered family; by such a family consisting of no more than  $N$  measures. For the second case the condition is (\*) whenever  $A_i \in \mathfrak{A}$ ,  $B_i \in \mathfrak{B}$ ,  $A_i \subseteq B_i B_{i+1}$  ( $i=1, \dots, n$ ),  $B_{n+1} = B_1$ , we have

$$\prod_{i=1}^n P(A_i|B_i) = \prod_{i=1}^n P(A_i|B_{i+1}).$$

The family  $\mathfrak{B}$  is quasi-additive if to each  $B_1, B_2 \in \mathfrak{B}$  there is at least one  $B \in \mathfrak{B}$  such that  $B_1 + B_2 \subseteq B$  and

$$P(B_1|B) + P(B_2|B) > 0.$$

If  $\mathfrak{B}$  is quasi-additive (in particular, additive) then (\*) is satisfied. The independence of the various axioms is shown by simple examples.

K. L. Chung.

**Thullen, Peter.** Über das Konvergenzproblem der relativen Häufigkeiten in der Wahrscheinlichkeitstheorie. Math. Ann. 131 (1956), 346-353.

A sequence  $(c_n)$  of real numbers is said to be quasi-convergent to the quasilimit  $c$  if, for any  $\epsilon > 0$ , the inequality  $|c_n - c| < \epsilon$  holds for "almost all"  $n=1, 2, \dots$ . Let  $(a_n)$  be a sequence of bounded real numbers, and denote by  $h_n$  the arithmetic mean of  $a_1, \dots, a_n$ . If the sequence  $(h_n)$  is quasicontvergent to some quasilimit  $h$ , it is proved that  $(h_n)$  converges in the ordinary sense to the limit  $h$ . In the particular case when all the  $a_n$  are equal to 0 or 1, this becomes a theorem on the asymptotic behavior of frequency ratios in a sequence of repeated experiments. Various further results concerning frequency ratios in successive sections of a sequence of experiments are given.

H. Cramér (Stockholm).

**Haáz, I. B.** Une généralisation du théorème de Simmons. Acta Sci. Math. Szeged 17 (1956), 41-44.

Let  $X$  be a binomial random variable with probabilities  $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$  and suppose that  $0 < p < 1$ .

Denote by  $P_1 = P(X \leq np-1)$ ,  $P_2 = P(X > np)$ , then  $P_1 > P_2$ . The author proves this theorem which is a generalization of a result of Simmons [Proc. London Math. Soc. 26 (1895), 290-323]. This generalization is however not quite new since Feldheim [Giorn. Ist. Ital. Attuari 10 (1939), 229-243; J. Math. Pures Appl. (9) 20 (1941), 1-16; MR 1, 246; 3, 1] obtained a stronger result which includes also an estimate of  $\Delta = P_1 - P_2$ .

E. Lukacs (Washington, D.C.).

**Suschowk, Dietrich.** Über Paare voneinander unabhängiger Ereignisse in kontinuierlichen und diskreten Wahrscheinlichkeitsfeldern. Arch. Math. 7 (1956), 221-224.

A probability distribution on the real line  $R$  is given. Subsets  $A, B, \dots$  of  $R$  are interpreted as events, with probabilities  $p(A), p(B), \dots$ . The events  $A$  and  $B$  are called non-trivially independent (n.t.i.), if  $p(A \cap B) = p(A)p(B)$  with  $0 < p(A) < 1$ ,  $0 < p(B) < 1$ . The following theorem is proved. a) In any distribution containing a not identically vanishing continuous component, it is possible to find a non-enumerable set of pairs of n.t.i. events. b) There are discrete (infinite as well as finite) distributions such that no pair of n.t.i. events exists. (In his proof of a) the author seems to assume that, if  $f$  is a continuous distribution function, there exists an interval  $I$  such that  $f$  is strictly increasing throughout  $I$ . However, the proof remains valid when this slip is corrected. The formula (3) of the paper, which is basic for the proof of b), is vitiated by a printer's error which is, however, easily corrected.)

H. Cramér (Stockholm).

**Broadbent, S. R.** Lognormal approximation to products and quotients. Biometrika 43 (1956), 404-417.

Let  $q = (x_1 x_2 \dots x_j) / (x_{j+1} \dots x_n)$  ( $1 \leq j \leq n$ ), where  $x_i$  is distributed either normally or in a rectangular distribution with small coefficient of variation  $\alpha_i$ . If all  $x_i$  are positive and independent as  $n \rightarrow \infty$  under general conditions, the distribution of  $\log q$  by the central limit theorem approaches a normal distribution. The author shows how to approximate the distribution of  $\log q$  by two methods using moments and compares these approximations with exact results in the case of a product and a quotient. He concludes that if the number of components are more than two, the lognormal approximation will give satisfactory results when the coefficients of variation are small and not too different. Tables make the applications simple.

L. A. Aroian (Culver City, Calif.).

**Lipschutz, Miriam.** On the magnitude of the error in the approach to stable distributions. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 281-287, 288-294.

Consider sums  $S_n = \sum_{k=1}^n X_k$  of independent, identically distributed random variables. W. Doeblin has given necessary and sufficient conditions for the convergence of the distribution functions of suitably normalized sums  $S_n$  to a stable distribution function [Studia Math. 9 (1940), 71-96; MR 3, 168]. The error of the approach was estimated by the reviewer in a general case. Here only positive continuous random variables are considered but the other conditions upon  $F(x)$  are rather weak. The

made assumptions and the given estimations of the error term naturally are complicated. Clearly the error term depends on the function  $d(m, x) = h(mx)/h(m) - 1$ , where  $1 - F(x) = xh(x)$  when the limiting stable distribution function is  $G_\gamma(x)$ ,  $0 < \gamma < 2$ . Corresponding to different assumptions about  $d(m, x)$  different error terms are deduced. Such error terms are given for finite  $x$ , and in the case  $0 < x < 1$  for  $x \rightarrow 0$ , and in the case  $1 < \gamma < 2$  for  $x \rightarrow \infty$ . The error term is obtained by the help of characteristic functions. The difficulty of computing the characteristic function of  $S_n$  for given  $h(x)$  can be carried through in the considered special case since for the convergence the major contribution to the sum is due to the maximum term.

H. Bergström (Göteborg).

**Maruyama, Gisirō.** On the Poisson distribution derived from independent random walks. Nat. Sci. Rep. Ochanomizu Univ. 6 (1955), 1-6.

Poisson distribution is derived as the limit distribution of the number of particles lying in an interval of a system of particles whose movements are independent of each other. Let  $Y_n(t) = x_n + X_n(t)$  represent the position of the  $n$ th particle at time  $t$  ( $n = 0, \pm 1, \pm 2, \dots$ ;  $t = 0, 1, 2, \dots$ ).  $X_n$  are random variables for which  $\{x_t - x_{t-1}, 2 = 0, \pm 1, \dots\}$  are mutually independent with common distribution function  $F(x)$ .  $X_n(t)$  are random variables for which  $\{X_n(t) - X_n(t-1), -\infty < n < +\infty, t \geq 0\}$  are mutually independent and obey the same distribution  $G(x)$  which is of non-lattice type.  $\{X_n(t), n = 0, \pm 1, \dots\}$  is independent of  $\{x_n, n = 0, \pm 1, \dots\}$ . Let  $N_I(t)$  be the number of  $n$ 's with  $Y_n(t)$  lying in the interval  $I = (a, b)$ . Then

$$\lim_{t \rightarrow \infty} \Pr\{N_I(t) = k\} = e^{-u} \frac{u^k}{k!}.$$

S. C. Moy (Detroit, Mich.).

**Dobrušin, R. L.** On Poisson's law for distribution of particles in space. Ukrain. Mat. Ž. 8 (1956), 127-134. (Russian)

Given a system of countably many particles moving on a line, let  $\eta_\Delta(t)$  be the number of particles in the interval  $\Delta$  (of length  $|\Delta|$ ) at time  $t$ . The particles will be said to move independently if the displacements between times 0 and  $t$  are independent of each other and of the initial positions, and have a common distribution. The distribution of the particles at time  $t$  is said to be Poisson with parameter  $\lambda \geq 0$  if  $\eta_\Delta(t)$  has a Poisson distribution with parameter  $\lambda|\Delta|$  and if  $\eta_{\Delta_1}(t), \dots, \eta_{\Delta_n}(t)$  are mutually independent when the intervals involved are nonoverlapping. The distribution is said to be generalized Poisson if  $\lambda$  is given a distribution in the preceding definition. The reviewer proved [Stochastic processes, Wiley, New York, 1953; MR 15, 445] that, if the initial distribution of particles is Poisson, the distribution remains so, with the same parameter, under independent motion of the particles. The author proves that if, for some constant  $\lambda \geq 0$ , (a)  $\lim_{|\Delta| \rightarrow \infty} M|\eta_\Delta(0)\Delta^{-1} - \lambda| = 0$ , and (b) for every  $l > 0$ ,  $\lim_{t \rightarrow \infty} \sum_n |P_t(l, n) - P_t(l, n-1)| = 0$ , where  $P_t(l, n)$  is the probability that a particle displacement between times 0 and  $t$  lies in the interval  $[nl, (n+1)l]$ , then the system distribution is asymptotically ( $t \rightarrow \infty$ ) Poisson, with parameter  $\lambda$ . The condition (b) is necessary if the conclusion is to hold for all initial distributions. Maruyama [see the paper reviewed above] proved a special case of this theorem. If (b) is kept but if (a) is replaced by the hypothesis that the initial distribution is invariant under translations, and that  $M\eta_\Delta(0) < \infty$  for all  $\Delta$ , it is proved

that then (a) holds, for some random variable  $\lambda$ , and that the system distribution is asymptotically generalized Poisson, where the parameter has the distribution of  $\lambda$ .  
J. L. Doob (Geneva).

★ **Hofmann, Martin.** Über zusammengesetzte Poisson-Prozesse und ihre Anwendungen in der Unfallversicherung. Stämpfli & Cie, Bern, 1955. 80 pp.

In this doctoral thesis the author investigates a process which consists in the occurrence of two distinct sequences of events. Mutual dependence of events belonging either to the same or to a different sequence is admitted. Denote by  $P(m, n; s, t)$  the probability that  $m$  events of the first sequence occurred during the time interval  $(0, s)$  and that  $n$  events of the second sequence occurred during  $(0, t)$ . The author considers the case of a bivariate compound Poisson process

$$P(m, n; s, t) = \int_0^\infty \int_0^\infty e^{-ks - lt} \frac{(ks)^m}{m!} \frac{(lt)^n}{n!} dU(k, l),$$

where  $U(k, l)$  is a bivariate distribution function defined over the positive quadrant. In the first part of the paper the author studies processes of this type and gives several bivariate extensions of univariate compound Poisson distributions, in particular of the negative binomial distribution. He also constructs a wide variety of univariate compound Poisson distributions and uses as his tool the following lemma which is of some independent interest: Let  $f(t)$  be a function which is completely monotone in  $(0, \infty)$  and suppose that  $g(t)$  is a function such that  $g'(t)$  is completely monotone in  $(0, \infty)$  then  $f[g(t)]$  is completely monotone on  $(0, \infty)$ .

The second part of the paper deals with accident insurance. The author examines available data and concludes that a compound Poisson process is suitable probabilistic model and that the hypothesis that the accident proneness varies from one insured person to the other is justified. The motivation for the study of his bivariate models is given by the necessity of distinguishing between occupational and non-occupational accidents. These two kinds of accidents correspond to the two distinct sequences of events.

E. Lukacs.

**Maruyama, Gisirō.** Fourier analytic treatment of some problems on the sums of random variables. Nat. Sci. Rep. Ochanomizu Univ. 6 (1955), 7-24.

Let  $X_1, X_2, \dots$  be identically distributed, independent, non-lattice random variables with non-negative mean  $m$ . Let  $S_n = X_1 + \dots + X_n$ . Several theorems are proved.

Theorem 1. Let  $N(a, a+h)$  be the expected number of sums  $S_n$  belonging to the interval  $(a, a+h)$ . Then

$$\lim_{a \rightarrow \infty} N(a, a+h) = \frac{h}{m}, \quad \lim_{a \rightarrow -\infty} N(a, a+h) = 0,$$

Theorem 2. Let  $m = 0$  and the variance of  $X$ , be  $\sigma^2$ . Let  $N_n$  be the number of sums  $S_\nu$ ,  $1 \leq \nu \leq n$ , belonging to the finite interval  $(-\frac{1}{2}h, \frac{1}{2}h)$ . Then

$$\lim_{n \rightarrow \infty} \Pr\{N_n \leq \frac{n^{\frac{1}{2}}h}{\sigma} x\} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^x e^{-\frac{1}{2}u^2} du, \quad x > 0$$

provided for some  $\delta > 0$ ,  $E(|X|^{3+\delta}) < \infty$ .

Theorem 3. Let  $N_n$  be the number of change of signs of the sums  $S_\nu$ ,  $1 \leq \nu \leq n$ . Then

$$\lim_{n \rightarrow \infty} \Pr\{N_n \leq \frac{\beta_1}{2\sigma} n^{\frac{1}{2}} x\} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^x e^{-\frac{1}{2}u^2} du,$$

where  $\beta_1 = E[|X_1|]$ .

Theorem 1 is a theorem by Chung and Pollard [Proc. Amer. Math. Soc. 3 (1952), 303-309; MR 14, 61] with the assumption  $\limsup_{|t| \rightarrow \infty} |\phi(t)| < 1$  removed where  $\phi(t)$  is the characteristic function of  $X_1$ . Theorem 3 was proved by Chung previously [Ann. of Math. (2) 51 (1950), 697-706; MR 11, 731].

The author used convolution transforms with Fejer's kernel. Using similar technique the following theorem concerning a stationary sequence is also proved.

Theorem 4. Let  $X, X, \dots$  be a Gaussian (real) stationary sequences with mean 0, autocorrelation coefficient  $\rho_k$  and spectral density  $f(\lambda)(2\pi)^{-1}$ ,  $-\pi \leq \lambda \leq \pi$ , continuous in a neighbourhood of  $\lambda=0$ ,  $\infty > f(0) \neq 0$ . Let  $N_n$  be the same as in Theorem 2 then

$$\lim_{n \rightarrow \infty} \Pr\left\{N_n \leq \frac{h}{\sqrt{f(0)}} n^{1/2} x\right\} = \left(\frac{2}{\pi}\right)^{1/2} \int_0^x e^{-u^2/2} du.$$

S. C. Moy (Detroit, Mich.).

Ikedai, Nobuyuki. Fluctuation of sums of independent random variables. Mem. Fac. Sci. Kyusyu Univ. Ser. A. 10 (1956), 15-28.

Let  $\mathcal{N}(Q)$  be the number of points in the set  $Q$ , let  $V_n$  be the stable distribution whose characteristic function is  $\exp(-|t|^\alpha)$ , and let  $S_n$  be the sum of  $n$  independent copies of some random variable  $X$  such that (1)  $\Pr(X \leq x)$  is not concentrated on an arithmetic progression and (2)  $\Pr(S_n/\text{const } n^{1/\alpha} \leq x) \rightarrow V_n(x)$  when  $n \uparrow +\infty$ .

Subject to certain extra restrictions on the basic distribution  $\Pr(X \leq x)$ , K. L. Chung and M. Kac [Mem. Amer. Math. Soc. no. 6 (1951), MR 12, 722] and G. Kallianpur and H. Robbins [Duke Math. J. 21 (1954), 285-307; MR 16, 52] have shown that if either  $\alpha \in [1, 2]$  and  $D_n = (1/2\alpha) \mathcal{N}(\{i: S_i \leq a, 1 \leq i\})$  for some  $a > 0$  or  $\alpha \in (1, 2]$  and  $D_n = (2/E(|X|)) \mathcal{N}(\{i: S_i > 0, S_{i+1} < 0, i \leq n\})$ , then

$$(3) \quad \Pr(D_n \leq m_n x) \rightarrow \int_0^x g_{1/\alpha}(u) du \text{ when } n \uparrow +\infty,$$

in which  $m_n$  is proportional to  $n^{1-1/\alpha}$  when  $\alpha > 1$  and to  $\log n$  when  $\alpha = 1$  and

$$g_{1-\gamma}(u) = (1/\pi\gamma) \sum_{n>0} (-u)^{n-1} \sin(n\pi\gamma) \Gamma(1+n\gamma)/n! \quad (\gamma \in (0, 1)), \\ = e^{-u} \quad (\gamma = 0).$$

Here, calculations based on a device of G. Maruyama [See the paper reviewed above] show that (1) and (2) alone  $\Rightarrow$  (3).

Corrections: the number of misprints is remarkable. Most are trivial, but the reader should note that the functions introduced in I. 13, p. 17, l. 4 p. 21, and I. 1, p. 22 have to be positive.

H. P. McKean, Jr.

Bendat, Julius. A general theory of linear prediction and filtering. J. Soc. Indust. Appl. Math. 4 (1956), 131-151.

The author presents a general theory of linear prediction and filtering as developed and applied by N. Wiener [Extrapolation, interpolation, and smoothing of stationary time series, Wiley, New York, 1949; MR 11, 118], L. Zadeh and J. Ragazzini [J. Appl. Phys. 21 (1950), 645-655; MR 12, 347], R. C. Booton [Proc. I.R.E. 40 (1952), 977-981], R. C. Davis [J. Appl. Phys. 23 (1952), 1047-1053; MR 14, 295] and J. Bendat [Northrop Aircraft, Inc., Rep. no. NAI-54-771 (1954)]. S. Kullback.

Gani, J. Sufficiency conditions in regular Markov chains and certain random walks. Biometrika 43 (1956), 276-284.

For positively regular Markov chains with a finite

number of states, transition probabilities of the form

$$p_{ij}(\theta) = \alpha_{ij} \exp\{K_{ij}\Lambda_1(\theta) + \lambda_2(\theta)\},$$

are known to admit a sufficient estimator of  $\theta$  in realizations of the chain starting with a fixed state and consisting of a fixed number of transitions. This paper considers whether transition probabilities of the same form will admit a sufficient estimator of  $\theta$  in other finite regular, but not positively regular, Markov chains. For chains with an irreducible subset of two or more states, in which a realization starts from a fixed state and consists of a fixed number of transitions, these probabilities are found to admit a maximum-likelihood estimator of the function  $g(\theta) = -\lambda_2'(\theta)/\Lambda_1'(\theta)$ , which is sufficient and unbiased. (From author's summary.) J. Wolfowitz.

Gani, J. 'Some theorems and sufficiency conditions for the maximum likelihood estimator of an unknown parameter in a simple Markov chain.' Biometrika, 42, 342-59. Biometrika 43 (1956), 497-498.

Correction of an error in the author's paper [Biometrika 42 (1955), 342-359; MR 17, 640] cited in the title.

J. Wolfowitz (Ithaca, N.Y.).

Blum, J. R.; and Rosenblatt, Murray. A class of stationary processes and a central limit theorem. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 412-413.

Let  $\eta = (\dots, \eta_{-1}, \eta_0, \eta_1, \dots)$  be a doubly infinite sequence of independent identically distributed random variables. Let  $T$  be the translation operator:

$$T\eta = \eta' = (\dots, \eta_0, \eta_1, \eta_2, \dots).$$

Let  $g = g(\eta)$  be a function defined on  $\eta$  for which  $E[g^2] < \infty$ . Let  $X_n = g(T^n \eta)$  ( $n = 0, \pm 1, \pm 2, \dots$ ). The following two theorems are announced. I. The spectral distributions function  $F(\lambda)$  of the process  $\{X_n\}$  is absolutely continuous. II. Let  $EX_n = 0$ . Let

$$X_{n,k} = E[X_n | \eta_k, \eta_{k+1}, \dots, \eta_{n+k-1}],$$

$$V_{n,k} = X_{n,k+1} - X_{n,k}, \text{ and } a_{s,k,k'} = E\{V_{s,k} V_{n+k,k'}\}, \text{ then} \\ \sum_{s=-\infty}^{\infty} \sum_{k,k'=1}^{\infty} |a_{s,k,k'}| < \infty \text{ and } \sum_{s=-\infty}^{\infty} \sum_{k,k'=1}^{\infty} a_{s,k,k'} \neq 0,$$

provided  $E\{|X_n|^\alpha\} < \infty$  for some  $\alpha > 2$ .

S. C. Moy (Detroit, Mich.).

Meinész, M. The problem of the gambler's ruin. Statistica, Neerlandica 10 (1956), 87-97. (Dutch. English summary)

By an application of the method of images the author obtains the expression

$$(*) \left[ \binom{n+m}{n} - \sum_{s=1}^{s=\max} \left\{ \binom{n+m}{n+a-sa-sb} + \binom{n+m}{m+b-sa-sb} \right. \right. \\ \left. \left. - \binom{n+m}{n-sa-sb} - \binom{n+m}{m-sa-sb} \right\} \right] p^n q^m$$

for the probability that in a random walk from  $(0, 0)$ , restricted to the points inside the barriers  $y = x + a$ ,  $y = x - b$ , the point  $(n, m)$  will be reached;  $p, q$  being the respective probabilities of a step in the positive  $x, y$  direction, ( $p + q = 1$ ). With the help of (\*) probabilities of ruin in a game between two gamblers with initial capitals  $a$  and  $b$  can easily be found. Some other applications and extensions of the author's method are considered.

H. A. David (Melbourne).

Conolly, B. W. Unbiased premiums for stop-loss reinsurance. Skand. Aktuarietidskr. 38 (1955), 127-134.

Vajda, S. Analytical studies in stop-loss reinsurance. II. Skand. Aktuarietidskr. 38 (1955), 180-191.

Earlier, Vajda [Skand. Aktuarietidskr. 34 (1951), 158-175] examined the problem of unbiased premiums for stop-loss reinsurance. He remarked that an unbiased estimator of the reinsurance premium depending upon one or more unknown parameters, might lead to a biased estimator when an unbiased estimate of the parameter or parameters is substituted into the correct formula. The question as to whether the substitution must necessarily lead to a biased premium was left open. If the method of premium calculation is to be unbiased, it is evident that the sample premium must have an expected value equal to the average amount which the reinsurance office must expect to pay in any year. Conolly tries a variety of numerical values under plausible hypotheses for a one-dimensional family of probability densities,  $k e^{-kx}$ , and shows that in each case a biased premium results. Vajda in this paper concludes on analytic grounds that the proposed method of premium calculation is necessarily biased in case that the probability density considered is normal and the reinsurance required exceeds zero (the only case of practical significance). Some other hypotheses lead asymptotically with increase in the number of insured to this same conclusion.

A. Bennett (Providence, R.I.).

Kracke, Helmut. Beiträge zur Prämienrückgewähr. Bl. Deutsch. Ges. Versicherungsmath. 3 (1956), 77-80.

Berger, Gottfried. Zur Frage des Verlaufs der Übersterblichkeit erhöhter Risiken. Bl. Deutsch. Ges. Versicherungsmath. 3 (1956), 57-75.

Dalcher, Andreas. Einige unstetige stochastische Prozesse. Z. Angew. Math. Phys. 7 (1956), 273-304.

In der Differentialgleichung  $dx/dt = x(x, t)$  sei die Zeit  $t$  die unabhängige,  $x$  die abhängige Variable. Verfasser betrachtet den folgenden stochastischen Prozess  $x(t)$ : in jedem Zeitintervall  $(t, t+dt)$  besteht eine Wahrscheinlichkeit  $w(x, t)dt$ , dass  $x$  einen Sprung ausführt. Wenn dieser zur Zeit  $t$  stattfindet und wenn die abhängige Variable in  $(t-0)$  den Wert  $y$  angenommen hat, so hat  $x$  unmittelbar nachher die Verteilung  $G(y, x, t)$ . Vom so erreichten

Punkt  $x$  aus folgt der Prozess bis zum nächsten Sprung der durch  $(x, t)$  gehenden Lösung der Differentialgleichung. Der Anfang des Prozesses sei durch die Verteilung  $F(x, 0)$  gegeben. Es ist  $F(x, t)$ , die Verteilung von  $x$  zu irgendeiner Zeit, zu berechnen. Die Lösungen werden auf zwei Arten erhalten. Mit Integralgleichungen gewinnt man einen besondern Aspekt und allgemeinere Ergebnisse des „Queueing“-Problems und ähnlicher Fragen. Durch die Laplace-Transformation lässt sich unter anderem das „Geräuschproblem“ behandeln. Hier wird die Störung jedoch als unstetig angenommen, während in der Literatur das Hauptgewicht auf den kontinuierlichen Fall gelegt wird. Anwendungen sind gegeben.

J. Wolfowitz (Ithaca, N.Y.).

Homma, Tsuruchiyo. On the many server queuing process with a particular type of queue discipline. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. 4 (1956), 90-101.

This paper deals with an  $s$ -server system with negative exponential output and arbitrary input whose mean is finite. A new arrival accepts service or joins the single queue, with probability  $p_i$ , where  $i$  is the total number of customers who are either in the queue or being served, when he arrives. It is assumed that  $p_0 = p_1 = \dots = p_{s-1} = 1$ , and  $p_i \geq p_{i+1} \geq 0$  for  $i \geq s$ . Let  $q'$  and  $q''$ , respectively, be the number of customers which each of two customers who consecutively join the process (by accepting service or joining the queue) finds ahead of him. Let

$$qq = P\{q' = j | q'' = i\}.$$

The author studies the character of the stochastic matrix  $\{qq\}$ , and gives sufficient and occasionally necessary conditions for it to be ergodic, null-recurrent, or transient.

J. Wolfowitz (Ithaca, N.Y.).

Ionescu, H. Two applications of probability calculus. Gaz. Mat. Fiz. Ser. A. 8 (1956), 225-242. (Romanian)

Some general remarks on normal and Poisson distributions and on Markov chains. One application is on congestion problems in telephoning and deduces Erlang's formula, the other is on sampling theory. O. Bottema.

See also: Cohn, p. 276; Bahvalov, p. 314; Sard, p. 322; Špaček, p. 330; Michalup, p. 336; Lyubotov, p. 346; Brücker-Steinkuhl, p. 367; Frishkopf, p. 367.

## STATISTICS

Guttman, Louis. "Best possible" systematic estimates of communalities. Psychometrika 21 (1956), 273-285.

At least four approaches have been used to estimate communalities that will leave an observed correlation matrix  $R$  Gramian and with minimum rank. It has long been known that the square of the observed multiple-correlation coefficient is a lower bound to any communality of a variable of  $R$ . This lower bound actually provides a "best possible" estimate in several senses. Furthermore, under certain conditions basic to the Spearman-Thurstone common-factor theory, the bound must equal the communality in the limit as the number of observed variables increases. Otherwise, this type of theory cannot hold for  $R$ . [Author's Summary.]

C. C. Craig (Ann Arbor, Mich.).

Cureton, Edward E. Rank-biserial correlation. Psychometrika 21 (1956), 287-290.

In the case of two correlated variables, let values of the one in a sample be known only by a ranking (ties are assigned mid-ranks) and values of the other only by a dichotomy (high and low). The author devises a measure of the correlation for such cases which may be regarded as a special case of both the Kendall  $\tau$  and the Spearman  $\rho$  coefficients.

C. C. Craig (Ann Arbor, Mich.).

Angoff, William H. A note on the estimation of non-spurious correlations. Psychometrika 21 (1956), 295-297.

The coefficient of correlation of the scores on a test  $t$  with those on a portion of the test is, at least in part, spurious. To estimate the non-spurious correlation, the

author supposes the subtest  $j$  replaced by a parallel test  $j'$  of equal effective length. Then  $r_{j'j}$  is not spurious and if in the expression for  $r_{j'j}$ ,  $r_{jj'}$  be estimated by the reliability  $r_{jj}$ , the result is proposed as an estimate of the non-spurious  $r_{jj}$ . In the case that  $j$  is a shortened parallel of the entire test the estimation formula assumes a very simple form.

C. C. Craig (Ann Arbor, Mich.).

**Reiersøl, Olav.** A note on the signs of gross correlation coefficients and partial correlation coefficients. *Biometrika* 43 (1956), 480-482.

Using the properties of matrices of definite forms, the author proves two theorems which have the following generalization: If the signs of all gross correlation coefficients of a set of variables may be made negative by a cogredient change of signs, or if the signs of all partial correlation coefficients of highest order of the set may be made positive by a cogredient change of signs, then, without any change of signs, any partial correlation coefficient of any order has the same sign as the corresponding gross correlation coefficient.

B. W. Jones.

**Patterson, H. D.** A simple method for fitting an asymptotic regression curve. *Biometrics* 12 (1956), 323-329.

The author considers the fitting of the Mitscherlich regression law for equally spaced ordinates, and derives simple ratio formulae which estimate the exponent with almost full efficiency.

P. Whittle (Wellington).

**Epstein, Benjamin.** Simple estimators of the parameters of exponential distributions when samples are censored. *Ann. Inst. Statist. Math.*, Tokyo 8 (1956), 15-26.

In a population with p.d.f.

$$f(x; \Theta) = e^{-x/\Theta} / \Theta \quad (x > 0, \Theta > 0)$$

observations in a sample of size  $n$  become available in the order  $x_1 \leq x_2 \leq \dots \leq x_n$ , but  $\Theta$  is to be estimated from the partial sample  $x_a, x_{a+1}, \dots, x_b$ . Then an unbiased estimator is

$$\Theta^* = [(a\beta_{a,n} + a - n)x_a + \sum_{j=a+1}^{b-1} x_j + (n - b + 1)x_b] / b,$$

where

$$\beta_{a,n} = 1 / \sum_{j=1}^a 1 / (n - j + 1).$$

This is said to be "almost best" and an estimate of the variance is given. The cases  $a=1$  and  $b=n$  are considered, as is the related problem of estimating  $A$  and  $\Theta$  when

$$f(x; \Theta, A) = e^{-(x-A)/\Theta} / \Theta \quad (0 \leq A \leq X < \infty, \Theta > 0).$$

A. S. Householder (Oak Ridge, Tenn.).

**Hammersley, J. M.; and Mauldon, J. G.** General principles of antithetic variates. *Proc. Cambridge Philos. Soc.* 52 (1956), 476-481.

The method of antithetic variates applied in the paper reviewed above is described more generally as follows: if

$$E[f(\eta_1, \eta_2, \dots, \eta_n)] = \theta,$$

where the  $\eta$ 's are random variables, one seeks a stochastic or functional dependence among the  $\eta$ 's in such a way that the precision of the estimate  $f$  of  $\theta$  is improved and the reliability is not sacrificed. The authors then confine their attention to the case

$$f(\eta_1, \dots, \eta_n) = f_1(\eta_1) + \dots + f_n(\eta_n).$$

Let  $F_j(y)$  be the cumulative marginal distribution function of  $\eta_j$ , with the inverse function

$$F_j^{-1}(x) = \inf_{F_j(y) \geq x} y \quad (0 \leq x \leq 1).$$

Then if  $g_j(x) = f_j[F_j^{-1}(x)]$ , the estimator is

$$t = \sum g_j(\xi_j).$$

A conjecture is then made, and, for  $n=2$ , is proved as a theorem: Let  $X$  denote the class of all functions  $x(z)$  which have the following two properties. (i)  $x(z)$  is a  $(1, 1)$ -mapping of the unit interval  $(0, 1)$  onto itself, and (ii) except perhaps for a finite set of points  $z$ ,  $dx/dz$  exists and equals 1. Then the conjecture is the following: If  $I$  denotes the infimum of  $\text{var } t$  when all possible stochastic or functional dependences between the  $\xi_j$  are considered, then, provided the  $g_j$  are bounded functions

$$\inf_{x_j \in X (j=1, 2, \dots, n)} \text{var} \sum_{j=1}^n g_j[x_j(\xi)] = I,$$

where the same  $\xi$  appears in each  $x_j(\xi)$ .

A. S. Householder (Oak Ridge, Tenn.).

**Klerk-Grobbe, Gerda; and Sandberg, H. D.** Confidence regions for the standard deviation of a normally distributed variate based on the mean range of a number of samples. *Statistica, Neerlandica* 10 (1956), 99-115. (Dutch. English summary)

Let  $\bar{w}_{m,n}$  be the mean of  $m$  ranges, each taken over  $n$  normal variates with standard deviation  $\sigma$ . The authors tabulate constants  $A_{m,n}$ ,  $B_{m,n}$  such that (\*)  $\Pr(A\bar{w} \geq \sigma) = 1 - \alpha$  and  $\Pr(B\bar{w} \leq \sigma) = 1 - \beta$ . If  $L = A/B$ , then

$$(**) \quad \Pr(A\bar{w} \geq L\sigma) = \beta;$$

thus for specified  $n$  and  $L$ , values of  $m$  may be obtained which approximately satisfy both (\*) and (\*\*). The constants  $A$ ,  $B$ ,  $L$  and  $L^{-1}$  have been calculated (for  $n=5, 7$  and various  $m, \alpha, \beta$ ) both with the help of normal and  $\chi$ -approximations to  $\bar{w}$ .

H. A. David (Melbourne).

**Quenouille, M. H.** Notes on bias in estimation. *Biometrika* 43 (1956), 353-360.

Let  $t_n$  be a statistic which estimates a statistical parameter  $\theta$ , in samples of size  $n$ , and satisfies the usual criteria of consistency, efficiency, and sufficiency (if a sufficient statistic exists). The author is interested in simple methods of reducing the bias in  $t$ . If

$$E(t - \theta) = a_1/n + a_2/n^2 + a_3/n^3 + \dots,$$

he shows that  $t'_n = nt_n - (n-1)t_{n-1}$  and  $t'_{2p} = 2t_{2p} - t_p$ , where  $n=2p$ , are biased to order  $1/n^2$  in place of  $1/n$ . Examples illustrate these results and an extension to the simultaneous estimation of two statistics is indicated. In order to maintain the efficiency of estimation it is suggested that  $t_{n-1}$  and  $t_p$  be used in place of  $t_n$  and  $t_p$  in the formulas for  $t'_n$  and  $t'_{2p}$ , where  $t_{n-1}$  is the average of estimates over all possible sets of  $n-1$  observations.

L. A. Aroian (Culver City, Calif.).

**Healy, W. C., Jr.** Two-sample procedures in simultaneous estimation. *Ann. Math. Statist.* 27 (1956), 687-702.

Let  $X_{ij}$  ( $i=1, \dots, k; j=1, \dots, n_k$ ) be N.I.D. with means  $\theta_i$  variances  $\sigma^2$ . Stein [Ann. Math. Statist. 16 (1945), 243-258; MR 7, 213] showed how to construct a confidence interval for  $\theta$  [i.e. for the case  $k=1$ ] of prescribed length and prescribed confidence coefficient. The

methods depends on a two-stage sample where the size of the second sample is determined by the observations in the first sample.

In the present paper the author shows how this procedure can be easily extended to construct simultaneous confidence intervals of prescribed lengths and prescribed confidence coefficients for all  $\theta_i$ , for all  $\theta_i - \theta_j$ , or for all normalized linear functions of the  $\theta_i$ . Multivariate extensions are also found. *D. G. Chapman* (Seattle, Wash.).

**Stuart, Alan.** Bounds for the variance of Kendall's rank correlation statistic. *Biometrika* 43 (1956), 474-477.

The object of this note is to obtain an upper bound for the variance of M. G. Kendall's rank correlation statistic  $t$  in terms of  $\tau = E(t)$  and the grade correlation coefficient  $\rho_s$ . The author informed the reviewer that inequality (8) and the main inequality (9) are incorrect (for instance, in the case of Kendall's inverse canonical ranking). In equation (20) the signs of  $\rho_s$  and  $\tau$  should be reversed in the case of an inverse canonical ranking. *W. Hoeffding*.

**David, F. N.; and Johnson, N. L.** Some tests of significance with ordered variables. *J. Roy. Statist. Soc. Ser. B.* 18 (1956), 1-20; discussion, 20-31.

In order statistics, most papers concern the estimation of parameters. Here, the testing of hypotheses is discussed and various tests are constructed, e.g. for the mean and the dispersion of a normal population.

The author gives a classification of more than one hundred papers on order statistics. *P. Johansen*.

**Noether, Gottfried E.** Two sequential tests against trend. *J. Amer. Statist. Assoc.* 51 (1956), 440-450.

Suppose that  $x_1, x_2, \dots$  are independent random variables whose deviations from their respective means are identically distributed. The author propose two tests for randomness (that is, equality of the means) against (i) a linear trend, (ii) "some irregular cyclical movement". In test (i) he chooses a positive integer  $j$ , puts  $y_m = 1$  or 0 according as  $x_m - x_{j+m}$  is positive or negative, and applies the appropriate sequential probability ratio test [cf. Wald, *Sequential analysis*, Wiley, New York, 1947, Ch. 5; MR 8, 593] to test the hypothesis that  $p = p_0 = \frac{1}{2}$ , where  $p$  is the parameter of the binomial distribution associated with the  $y$ 's, against an alternative  $p = p_1$  (corresponding to a linear trend of the  $x$ 's). In test (ii) he puts  $y_m = 1$  or 0 according as the sequence  $x_{3m}, x_{3m+1}, x_{3m+2}$  is monotonic or not, and proceeds similarly. He discusses suitable choices of  $j$  and  $p_1$  and provides tables to facilitate use of the tests. He claims that the expected number of observations required in test (i) is smaller than that needed in other tests in current use.

*H. P. Mulholland* (Birmingham).

**David, Herbert T.; and Kruskal, William H.** The WAGR sequential  $t$ -test reaches a decision with probability one. *Ann. Math. Statist.* 27 (1956), 797-805.

The abbreviation WAGR refers to Wald, Arnold, Goldberg and Rushton, who developed this test. To the references in the paper Arnold's introduction to "Tables to facilitate sequential  $t$ -tests" [Nat. Bur. Standards Appl. Math. Ser., no. 7, Washington, D.C.; MR 13, 141] can be added. *D. M. Sandelius* (Göteborg).

**Higuti, Isao.** Note on the sums of the independent variates of K. Pearson's type V. *Ann. Inst. Statist. Math., Tokyo* 8 (1956), 55-59.

Let  $X_1, \dots, X_n$  be independently distributed with

density function  $f(x; s_i + \frac{1}{2}, \gamma_i)$  ( $i=1, \dots, n$ ), where  $f(x; \lambda, \gamma) = \gamma^\lambda x^{-(\lambda+1)} e^{-\gamma/x} / \Gamma(\lambda)$  for  $x \geq 0$ . The density of  $Y = (\sum_{i=1}^n \sqrt{\gamma_i})^2 \sum X_i$  is obtained (without proof) as a weighted sum of densities of the type  $f(x; \lambda_\mu, 1)$ . The weights are explicitly designated for some special cases in which the variates are identically distributed.

*I. Olkin* (East Lansing, Mich.).

**Akaike, Hirotugu.** On the distribution of the product of two  $\Gamma$ -distributed variables. *Ann. Inst. Statist. Math., Tokyo* 8 (1956), 53-54.

Let  $X_1, X_2$  be independently distributed with density function

$$\alpha_i \lambda_i e^{-\alpha_i x} x^{\lambda_i - 1} / \Gamma(\lambda_i) \text{ for } x \geq 0 \quad (i=1, 2).$$

The distribution function of  $X = X_1 X_2$  in the case  $\lambda_2 - \lambda_1 = n + \frac{1}{2}$  for integral  $n \geq 0$ , is found to be

$$P(X \leq t) = \sum_{i=0}^n c(i, \lambda_1, \lambda_2) I(2\sqrt{(\alpha_1 \alpha_2 t)}, \lambda_1 + \lambda_2 - i - \frac{1}{2}),$$

where  $I(v, p)$  is the incomplete gamma function.

*I. Olkin* (East Lansing, Mich.).

**Baitsch, Helmut; und Bauer, Rainald K.** Zum Problem der Merkmalsauswahl für Trennverfahren (Barnard-Problem). *Allg. Statist. Arch.* 40 (1956), 160-167.

The authors consider the discrimination problem of  $n$ -variate analysis and in particular the question as to whether the discrimination procedure should employ all  $n$  characteristics or whether the use of a subset of  $m < n$  variates is sufficient. For samples in which the observations come from paired individuals they propose the following procedure: Denote the corresponding  $n$  variates in the two groups by  $(x_1, x_2, \dots, x_n)$ ,  $(y_1, y_2, \dots, y_n)$ . Compute the  $n$  simple correlation coefficients  $r(x_i y_i)$  ( $i=1, 2, \dots, n$ ). If the  $j$ th characteristic yields the largest statistically significant correlation, compute the  $(n-1)$  partial correlation coefficients  $r(x_i y_i \cdot x_j y_j)$  ( $i=1, 2, \dots, n, i \neq j$ ). If the  $k$ th characteristic yields the largest statistically significant partial correlation compute the  $(n-2)$  partial correlation coefficients  $r(x_i y_i \cdot x_j y_j \cdot x_k y_k)$  ( $i=1, 2, \dots, n, i \neq j, k$ ). Continue this procedure until no statistically significant correlations are obtained. The discriminatory variates are then those which have yielded statistically significant correlations. The partial correlations are to be computed by the use of Yule's recursion formula

$$r(xy \cdot z \dots vw) = \frac{r(xy \cdot z \dots v) - r(xw \cdot z \dots v)r(yw \cdot z \dots v)}{[(1 - r^2(xw \cdot z \dots v)) \cdot (1 - r^2(yw \cdot z \dots v))]^{\frac{1}{2}}}$$

The procedure is illustrated by an application to a 7-variate set of fingerprint characteristics from 50 pairs of mother, child. *S. Kullback* (Washington, D.C.).

**Cox, C. P.** A geometrical derivation of the analyses of covariance and variance. *J. Roy. Statist. Soc. Ser. A.* 119 (1956), 333-335.

**Tukey, John W.** Variances of variance components. I. Balanced designs. *Ann. Math. Statist.* 27 (1956), 722-736.

Variances of variance components for random samples from finite populations are presented. The cases considered include balanced single and double classifications, Latin Squares, and balanced incomplete blocks. The author derives his results from his "polykays" [same Ann. 27 (1956), 37-54; MR 17, 868]. *M. Muller*.

Searle, S. R. Matrix methods in components of variance and covariance analysis. *Ann. Math. Statist.* 27 (1956), 737-748.

The author calculates, by matrix methods, the sampling variances of the least square estimates of the components of variance in an unbalanced one-way classification, and of the components of variance in the two-variable case. The methods are also used to obtain the large sample dispersion matrix of the maximum likelihood estimates of the components of variance and covariance.

P. Whittle (Wellington).

Royston, Erica. Studies in the history of probability and statistics. III. A note on the history of the graphical presentation of data. *Biometrika* 43 (1956), 241-247. Part I was listed in MR 16, 781 and Part II in MR 17, 931.

Williams, C. B. Studies in the history of probability and statistics. IV. A note on an early statistical study of literary style. *Biometrika* 43 (1956), 248-256. For Parts I, II, III see above.

Gilbert, Edgar J. The matching problem. *Psychometrika* 21 (1956), 253-266.

The matching problem is that of two decks of cards which are of several kinds, the number of each kind not necessarily the same in both decks, with a match occurring when cards of like kind are in the same position. Chief attention is paid to the case where each deck contains cards of  $s$  kinds and  $c$  of each kind, specification ( $c^s$ ), as in extra-sensory perception experiments. A table is given for the probability of at least  $h$  matches to five decimals for  $c=1, 2, s=2(1)11$ ;  $c=3, s=2(1)8$ ;  $c=4, 5, s=2(1)5$ ;  $s=2, c=6(1)12$  and  $s=3, c=6, 7$ . A considerable discussion is given to approximations of these distributions without recognition of the asymptotic result of Cattaneo [*Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat.* 101 (1942), 89-104; MR 8, 247].

J. Riordan (New York, N.Y.).

See also: Dupač, p. 336; Broadbent, p. 340; Conolly, p. 343; Vajda, p. 343; Claringbold, p. 367; Katz and Wilson, p. 367; Garner and McGill, p. 367; Sapožkov, p. 368; Tihonov, p. 368.

## PHYSICAL APPLICATIONS

### Mechanics of Particles and Systems

Válcovici, Victor. Sur une extension des principes variationnels de la mécanique et sur l'existence d'autres principes analogues. *C. R. Acad. Sci. Paris* 243 (1956), 1096-1098.

E. Storch [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 162-167; MR 17, 1217] has shown that Hamilton's principle for the conservative systems remains valid also for a large class of asynchronous motions, i.e. when the time is varied.

The author of this paper starts from the general variational mechanical principle

$$\int_{t_0}^{t_1} (\delta L + \delta' T + 2T \frac{d}{dt} \delta t) dt = 0$$

for asynchronous motions, where  $\delta L$  denotes the virtual work of the forces, and the  $\delta' T$  the variation of the kinetic energy. He declares then that from this principle are deducible not only Hamilton's principle and the principle of the least action but also an infinity of other variational principles which all ought to be valid for non-holonomic systems of a very general shape (not only those with linear dependence of the velocities). The note is very brief and contains in reality only assertions, so it remains to wait for the full exposition of these results to be fully convinced.

T. P. Andelić (Belgrade).

Lyubotov, Yu. V. Application of the principle of virtual displacements to the calculation of accidental errors in the position of a mechanism. *Trudy Inst. Mašinoved. Sem. Točn. Mašinostro. Priborostr.* 8 (1955), 3-17. (Russian)

The author treats the mechanism without flexible chains and kinematic rotational pairs as a material system with stationary constraints

$$f_{\sigma}(q_{\sigma}) \geq 0 \quad (\sigma = 1, 2, \dots, s),$$

where  $q_{\sigma}$  are parameters which determine the proportions of the mechanism, or  $f_{\sigma}(q_{\sigma}) = 0$ ;  $q_{\sigma_0}$  is the ideal value of  $q_{\sigma}$ . Let  $\theta_{\alpha}$  ( $\alpha = 1, 2, \dots, a$ ) be the generalized coordinates of

the drivers and  $\varphi$  one coordinate of the pulled element, then  $\varphi = \varphi(\theta_{\alpha}, q_{\sigma})$ . Acting in the ideal equilibrated position of the system ( $A_0$ ), with the unit generalized force ( $\phi = 1$ ) on the pulled element and with the generalized forces  $\Theta_{\alpha}$  on the drivers, one obtains by means of the principle of the virtual displacements the differential relations among the generalized coordinates and forces

$$\partial \varphi / \partial \theta_{\alpha} = -\Theta_{\alpha}, \quad \partial \varphi / \partial q_{\sigma} = -Q_{\sigma},$$

where  $Q_{\sigma}$  is the generalized reaction of the constraint. These relations represent the scalar errors in the position of a mechanism. The original errors can be, further, presented as free or sliding vector (for translation  $\Delta y = y^0 \Delta y$ , and for rotation  $\Delta v = v^0 \Delta v$ ) and the reactions in the kinematic pair are  $\partial \varphi / \partial y = (Ry^0)$ ,  $\partial \varphi / \partial v = (Lv^0)$ , where  $y^0, v^0$  are unit vectors. The case of the plane vector error is treated. It is shown that the transmission ratio can be expressed as the product of two quantities: 1) the non-accidental part ( $Z$ ) and 2) the quantity, which depends on the accidental parameter. The mean value of the accidental transmission ratio is zero, but the dispersion is  $\frac{1}{2} Z^2$ . Four examples are discussed to illustrate the theory (mechanism for multiplication, mechanism with eccentric thumb, mechanism for the relation  $\varphi = \arctg(xz_0^{-1} \sin \theta)$ , and the conoid mechanism).

D. Rašković.

Plainevaux, J. E. Recherche du profil à donner aux engrenages coniques droits pour que leur rapport de réduction reste constant et indépendant de l'angle que les axes forment entre eux. *Acad. Roy. Belg. Bull. Cl. Sci.* (5) 42 (1956), 854-860.

Le profil d'une roue conique est l'intersection par une sphère dont le centre coïncide avec le sommet du cône primitif. Par des considérations de cinématique et de géométrie l'auteur démontre que, quel que soit l'angle des axes choisi pour le fonctionnement, le rapport des vitesses angulaires ne reste constant que si chacun des profils est tel que ses grands cercles normaux sont à distance constante du point de percée de l'axe dans la sphère; chaque profil est donc une développante de cercle sphérique.

O. Bottema (Delft).

Vitenzon, I. G. On the relative motion of a material point with variable mass. Har'kov. Gos. Univ. Uč. Zap. 29=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 21(1949), 87-99. (Russian)

The author remarks that, as a rule, works devoted to the dynamics of bodies with variable mass study the motion of such a body relative to a Galilean inertial system of reference. Exceptionally, problems involving two bodies with variable mass are studied, with the emphasis on the relative motion of two interacting particles. Even in these investigations, however, use is not made of the general theory of the relative motion of variable masses but of an equation which, as is shown later, has only a limited field of application in the case of bodies with variable mass.

The kernel of the paper is § 2, which is devoted to the derivation of general differential equations for the relative motion of a particle with variable mass, i.e. its motion relative to an arbitrary non-inertial coordinate system, and to several applications of these equations to general problems of two bodies.

In § 3 these results are specialized to give the differential equations of the relative motion of a free particle of variable mass in the gravitational field of the rotating Earth. In the derivation of these equations the idea of weight must be re-examined — the weight of a particle of variable mass  $m(t)$  is

$$p = m(t)g + u^* \frac{dm}{dt},$$

where  $u^*$  is the velocity, relative to the Earth, of the 'variable portions' of the particle (i.e. those portions responsible for the change of mass). The equation of motion of the particle is then

$$m(t)w^* = b + p - v^* \frac{dm}{dt} - 2m(t)w \wedge v^*,$$

where  $v^*$ ,  $w^*$  are, respectively, the velocity and acceleration of the particle relative to the Earth,  $b$  is the resisting force, and  $w$  the angular velocity of rotation of the Earth.

Cases in which the particle behaves like one with constant mass are considered.

In § 4 these equations are applied to the problem of a spherical evaporating drop falling through a resisting medium from zero initial velocity. The assumption is made that the centre of mass of the 'variable portions' is that of the whole drop (at the instant of evaporation), which is the case when evaporation occurs with spherical symmetry so that the drop remains spherical. The velocity at any time is found for the cases in which resistance is proportional to the velocity and to the square of the velocity. In the first case are found the maximum velocity, limiting velocity (i.e. velocity when evaporation is complete) and, hence, maximum distance of travel of the drop. In the second case the limiting velocity is found.

Lastly, the equations of § 3 are applied to the case of a particle of variable mass projected into space from a point in the Northern hemisphere, in the plane of the meridian, at a definite initial velocity, making a definite angle with the horizontal. Necessary and sufficient conditions are given for the particle 1) to describe a straight line with constant velocity relative to the Earth; 2) to describe a parabola as would a body of constant mass.

R. A. Rankin and J. Burlak.

Kolsrud, Marius. Variation principles depending on the constants of motion of the mechanical system. Arch. Math. Naturvid. 53 (1956), 183-192.

A class of variational principles is derived which generalizes Hamilton's principle and the variational principles given by Schieldrop [Proc. Cambridge Philos. Soc. 51 (1955), 469-475; MR 16, 1167]; they are obtained by adding to a multiple of the Lagrangian a total time derivative and a function built up from the solution of the Hamilton-Jacobi differential equation. The relativistic case is also treated.

H. D. Block (Ithaca, N.Y.).

Isilinskii, A. Yu. On relative equilibrium of a physical pendulum with a movable fulcrum. Prikl. Mat. Meh. 20 (1956), 297-308. (Russian)

The problem of relative equilibrium of a rigid body with a movable point of support is discussed analytically in the case of a physical pendulum.

E. Leimanis.

Sponder, Erich. Eine genäherte Behandlung des schweren symmetrischen Kreisels in nicht-Eulerschen Koordinaten. Z. Angew. Math. Phys. 6 (1955), 462-478.

Let  $OXYZ$  be a fixed frame ( $OX$ -vertical), and  $Oxyz$  one bound to a heavy symmetric top ( $Ox$  being the axis of symmetry). The non-Eulerian angles introduced (with appropriate orientations) are:  $\alpha$  between  $Ox$  and  $OXY$ ,  $\beta$  between  $OX$  and  $OxZ$ , and  $\psi$  between  $Oy$  and the line of nodes. The components of the angular velocity on  $Oxyz$  are expressed in terms of these angles and their derivatives and the expressions become the same as in the case of the Euler angles after some sign changes and transpositions of the symbols "sin" and "cos". In the case of finite motions, the angle of nutation must be introduced to write the equations of motion, for which purpose Lagrange's equations are used. The bulk of the paper deals with small oscillations [see Routh, Advanced rigid dynamics, 6th ed., Macmillan, London, 1930, art. 214, where symmetry is not assumed]. The paper works with the complex variable  $\gamma = \alpha + i\beta$  (in this approximation  $\alpha$  and  $\beta$  become the angles of  $Ox$  with  $OX$  and  $OY$ ) for which the equation  $\ddot{\gamma} - i\dot{\gamma} + c\gamma = 0$  is obtained (in properly chosen time units),  $c$  being a real constant. If in the last equation  $i$  is replaced by  $-a + i$ ,  $a$  real, viscous damping is introduced. Let  $N$  and  $\Omega$  be, respectively, the circular frequencies of nutation and precession of the undamped motion. Then, for small  $a$ , the corresponding logarithmic decrements are  $-Na/(N - \Omega)$  and  $\Omega a/(N - \Omega)$ . The author stresses the need for an error estimate here, and compares the sum and product of the two complex frequencies with their respective exact values ( $-a + i$  and  $c$ ). The sum is exact, the product is in relative error of  $a^2/(1 + 4c)$  which does not settle the question of the error in each separate frequencies. The approximations to the real frequencies of nutation and precession are not affected by damping.

A. W. Wundheiler (Chicago, Ill.).

Jeffreys, Harold. A modification of Lagrange's equations for small oscillations when some natural frequencies are high. Quart. J. Mech. Appl. Math. 9 (1956), 247-248.

A method is shown how to treat holonomic conservative systems when some of the generalized velocities  $\dot{q}_i$  ( $i = 1, 2, \dots, n$ ) are absent from the Lagrangian function  $L(q_i, \dot{q}_i, t)$ , respectively when some of these velocities are negligible as in the case of small oscillations where some natural frequencies are high.

T. P. Andelić.

- ★ **Barkhausen, H.** Einführung in die Schwingungslehre, reibt Anwendungen auf mechanische und elektrische Schwingungen. S. Hirzel Verlag, Leipzig, 1956. vii+128 pp. DM 4.00.

Dieses Buch enthält die Beiträge zu dem „Handwörterbuch der Naturwissenschaften“, die ich erstmalig 1911 für 1. Auflage und 1932 entsprechend umgearbeitet für dessen 2. Auflage verfasst habe.

*Aus dem Vorwort zur 2. Auflage.*

- Hesse, H.** Strömung in Blasleitungen. Ing.-Arch. 24 (1956), 299–307.

- Breus, K. A.** Inertial unsteady motion of a solid cylinder. Dopovidi Akad. Nauk Ukrain. RSR 1956, 321–324. (Ukrainian. Russian summary)

The author studies the deformation of a solid cylinder subject to the striking action of a rigid plate at the top. The problem is reduced to the solution of the Laplace equation, with specified boundary-conditions at the top, bottom and on the surface. Numerical methods are employed to the specific problem. (The author's analysis is, however, rather scanty; important details are lacking.)

*K. Bhagwandin (Oslo).*

- Oswald, Telford W.** Dynamic behavior during accelerated flight with particular application to missile launching. J. Aero. Sci. 23 (1956), 781–791.

The author considers the equations of motion of a missile in which lift and damping forces are included. The longitudinal motion is assumed to be uniform acceleration whereas all other motions are assumed to be small. The forces and moments produced by the control surfaces and thrust misalignment are treated as forcing functions. The six equations of motion are thus linear with variable coefficients and can finally be reduced to a single ordinary differential equation. An exact solution is given in terms of integrals of the Fresnel type which depend on a single parameter only, the relative damping.

The initial conditions and solution are worked out for launching from rest and for finite launch speed. The effect of cross-wind is considered and a typical example is given.

*G. H. Handelman (Troy, N.Y.).*

See also: Bauer, p. 308; Watanabe, p. 317; Steiner, p. 328; Toupin, p. 349; Czaykowski, p. 352; Cohn, p. 356; Faure, p. 361.

### Statistical Mechanics

- Teramoto, Ei.** The statistical mechanical aspect of the *H*-theorem. II. Progr. Theoret. Phys. 15 (1956), 480–486.

This paper analyzes further a one-dimensional hard sphere model for which the approach to equilibrium was studied earlier by the author in collaboration with Suzuki [same journal 14 (1955), 411–422; MR 17, 812]. The Boltzmann *H*-function is defined both for fine-grained and coarse-grained cells of phase space, and it is shown how its approach to the equilibrium value is the more rapid, the larger the cells are. *L. Van Hove (Utrecht).*

- Ono, Syu.** A note on the variation principle in the kinetic theory of gases. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 5 (1955), 87–96.

It is known that the first term in the Chapman-Enskog

expansion for a mixture of dilute gases satisfies a variation principle. This variation principle is here given the following physical interpretation. Every deviation from the equilibrium distribution function gives rise to two kinds of irreversible entropy production. The first one is due to the variation of the distribution function in space. The second kind is due to the local deviation from the Maxwell-Boltzmann distribution. Consider those deviations for which the second kind of entropy production vanishes. The one that maximizes the rate of entropy production is equal to the first term in the Chapman-Enskog expansion. *N. G. van Kampen (Utrecht).*

- ★ **Oswatitsch, Klaus.** Gas Dynamics. English version by Gustav Kuerti. Academic Press Inc., New York, 1956. xv+610 pp. \$12.00.

A translation of the German original reviewed in MR 14, 814.

- Burgers, W. G.** Geometric description of dislocations. Nederl. Tijdschr. Natuurk. 22 (1956), 245–270. (Dutch)

Dislocations form a certain class of irregularities in the grids of crystalline solids and are of great use for understanding a number of phenomena which these solids display (e.g., the appearance of slip-lines and slip-planes in a crystal under deformation, and the discrepancy between the observed and calculated strength of a crystal under stress).

While it is fairly simple to visualise the most elementary dislocations, the situation becomes more complex when several dislocations are present. Moreover, a dislocation may travel through the crystal lattice, and when several dislocations meet or intersect, complicated configurations are generated which are difficult to visualise. The author tries to clarify some of these situations. Following the most elementary screw- and edge-dislocations and their characterization by a dislocation line and a Burgers vector, he discusses mixed dislocations (partly screw- and partly edge-) and the possibility of decomposition of dislocations. Next comes the motion of a dislocation through the lattice, and in particular the case where two perpendicular screw-dislocations intersect. The last part of the paper is concerned with the more complicated Shockley dislocation and how this enables the close-packed hexagonal lattice to be gradually transformed into the close-packed cubic lattice [cf. Melmore, Nature, 159 (1947), 817; MR 9, 53], a possibility which may be of importance for the study of allotropic modifications. The discussion is accompanied by 25 nice diagrams.

*J. A. Steketelee (Toronto, Ont.).*

See also: Akutowicz, p. 304; Memoirs of the unifying study. . . , p. 332; Truter, p. 339; McCrea, p. 355; Uhlhorn, p. 360; Klimontovic, p. 360.

### Elasticity, Visco-Elasticity, Plasticity

- Hemp, W. S.** Fundamental principles and theorems of thermo-elasticity. Aero. Quart. 7 (1956), 184–192.

Expository paper treating the linear theory only.

*W. Noll (Pittsburgh, Pa.).*

- Love, E. R.** Linear superposition in visco-elasticity and theories of delayed effects. Austral. J. Phys. 9 (1956), 1–12.

This paper concerns one-dimensional linear stress-

strain relations of the form

$$(1) \quad s(l) = \int_0^l \sigma(l-\omega) d\psi(\omega)$$

( $s$ =stress,  $\sigma$ =strain). The author gives three conditions and proves that these, together with linearity, characterize all relations of the form (1). The conditions are phrased in physical terms in order to allow direct experimental verification. A similar analysis is given in the case where the upper limit of the integral in (1) is replaced by  $\infty$ .

W. Noll (Pittsburgh, Pa.).

**Ericksen, J. L.; and Toupin, R. A.** Implications of Hadamard's conditions for elastic stability with respect to uniqueness theorems. *Canad. J. Math.* 8 (1956), 432-436.

This paper is concerned with what has been called "The main problem of finite elasticity" [Truesdell, *Z. Angew. Math. Mech.* 36 (1956), 97-103; MR 18, 162], i.e. the problem of finding the conditions on the strain energy function necessary to exclude physically unacceptable behavior. Specifically, the authors investigate the connection between the stability of a finite deformation with the uniqueness of solutions of boundary value problems for superimposed infinitesimal deformations. Following Hadamard [Leçons sur la propagation des ondes et les équations de l'hydrodynamique, Hermann, Paris, 1903] they call a finite deformation stable (neutrally stable) if the second variation of the total strain energy is positive (non-negative). Results: Stability implies uniqueness only for displacement boundary value problems but not for stress boundary value problems. Neutral stability fails to imply uniqueness in either case. These results indicate that for finite deformations, Hadamard's definition of stability is not in accordance with the intuitive notion and that refinement is desirable. W. Noll.

**Toupin, Richard A.** The elastic dielectric. *J. Rational Mech. Anal.* 5 (1956), 849-915.

The author has performed a real service by giving a clear, self-contained presentation of a general theory of elastic dielectrics, clarifying many points which have been sources of error or confusion to workers in this field. Some of the discussion, such as the elegant treatment of two-point tensor fields or the section on deformation and rotation are of equal interest to workers in other areas of continuum mechanics.

A brief description of Lorentz's calculation of the electrostatic field of a uniform lattice of point dipoles is included as partial motivation for introducing a local electric field in addition to the Maxwell field. Otherwise, the treatment is purely field theoretic. Constitutive equations are deduced with and without the aid of a variational principle. The author decisively refutes a common assertion that proper balance of moments cannot be attained with stresses derived from a strain energy function, noting that moments balance if and only if the strain energy satisfies a simple invariance requirement. Some pains have been taken to devise and explain a scheme for decomposing the forces acting into stresses and body forces of local and non-local character. This contributes considerably to the clarity and simplicity of the entire treatment.

The invariance requirements imposed by the condition of material isotropy are analyzed in considerable detail. For such materials, general solutions for simple shear of a homogeneously polarized infinite slab and for a

homogeneously deformed and polarized ellipsoid are derived. One novel feature is a quantitative explanation of the contribution of the local electric field to the electrostrictive effect. A general scheme for treating other types of material symmetry is briefly outlined, after which the author discusses approximate theories. One point of interest is that, in certain media, the stress may depend on rotations as well as strains when the deformation is infinitesimal and the polarization sufficiently strong.

J. L. Ericksen (Washington, D.C.).

**Radok, J. R. M.** On the solution of problems of dynamic plane elasticity. *Quart. Appl. Math.* 14 (1956), 289-298.

The author introduces the theory of functions of a complex variable into dynamic plane elasticity in a manner similar to Muskhelishvili's procedure in the static plane elasticity of isotropic media. The widened Airy's stress function ( $U$ ) must be the solution of a generalized bi-harmonic equation

$$(1) \quad \{\Delta - \rho(\lambda + 2\mu)^{-1} \partial^2 / \partial t^2\} \{\Delta - \rho\mu^{-1} \partial^2 / \partial t^2\} U = 0,$$

with the condition

$$\{\Delta - \rho\mu^{-1} \partial^2 / \partial t^2\} \{\tau_{xy} + \partial^2 U / \partial x \partial y\} = 0.$$

Using the transformation  $\xi = x + ct$ ,  $\eta = y$ , the real solution of Eq. (1) is  $U = \sum F_j(z_j) + \sum \bar{F}_j(\bar{z}_j)$ ,  $j=1, 2$ ,  $z_j = \xi + s_j \eta$ ,  $s_j = i f_j(\rho, c, \lambda, \mu)$ , where  $F_j$  are analytic functions of the complex variables  $z_j$ . Expressing the stresses and the displacements in terms of the derivatives of  $F_j$ , the boundary value problems of dynamic elasticity are reduced to the problems of complex function theory. The obtained stress equations are the known Titchmarsh's relations [Introduction to the theory of Fourier integrals, Oxford, 1937] deduced by means of Hilbert's transforms. To illustrate the advantages of this theory three problems are solved: the moving Griffith crack [Yoffe's solution, *Phil. Mag.* (7) 42 (1951), 739-750; MR 13, 302], the parabolic punch moving over a half-plane [Galin's solution by a similar method, "Contact problems of the theory of elasticity", Gostekhizdat, Moscow, 1953; MR 16, 644] and moving dislocation [Eshelby's solution, *Proc. Phys. Soc. Sect. A.* 62 (1949), 307-314].

D. P. Rašković (Belgrade).

**Žitkov, P. N.** A plane problem of the theory of elasticity of a nonhomogeneous orthotropic body in polar coordinates. *Trudy Voronezh. Gos. Univ. Fiz.-Mat. Sb.* 27 (1954), 20-29. (Russian)

The author finds solutions for problems of axial symmetry when the elastic constants of the orthotropic material are functions of the radius. In one particular case, the Airy stress function is determined by a hypergeometric equation which is analogous to an equation deduced by S. A. Čaplygin in his paper on gas jets [Ud. Zap. Imp. Moskov. Univ. Otd. Fiz.-Mat. 21 (1904)].

J. R. M. Radok (Oakland, Calif.).

**Kovalenko, A. D.** Bending of circular plates of variable thickness. *Akad. Nauk Ukrain. RSR. Prikl. Meh.* 1 (1955), 154-176. (Ukrainian. Russian summary)

The problem of the title is treated under the assumption that the normal pressure can be expanded in a Fourier cosine series,

$$q(r, \theta) = q_0(r) + \sum_{k=1}^{\infty} q_k(r) \cos k\theta.$$

The partial differential equation for the deflection then becomes an ordinary differential equation in  $r$  with parameter  $k$ . For  $k=0$  and 1, and for particular choices of the form of the flexural rigidity  $D$  as a function of  $r$ , some rather involved substitutions lead to a solution of the differential equation in terms of hypergeometric functions. For  $D$  proportional to the radius, a solution is given for arbitrary  $k$ . When the thickness is proportional to the radius, an approximate solution is given for the deflections for a freely vibrating plate. No particular boundary conditions are considered.

H. P. Thielman and H. J. Weiss.

Eringen, A. C. New numerical results of the theory of buckling of sandwich cylinders. *J. Appl. Mech.* 23 (1956), 476-477.

A formula for the critical buckling stress of a sandwich cylinder has been given previously by the author [*J. Appl. Mech.* 73 (1951), 195-202; MR 13, 887]. An IBM card program has been developed to evaluate this result and two sets of minimum critical stresses are given, in one case the faces rippling in the same direction and in the other, the opposite. Modifications for stresses beyond the proportional limit and corrections to the previous paper are included.

G. H. Handelman (Troy, N.Y.).

Wittmeyer, H. Einfache angenäherte Berechnung der Biegeeigenfrequenzen eines einseitig eingespannten Balkens ungleichförmigen Querschnittes, sowie der Eigenwerte ähnlicher Variationsprobleme. *Z. Angew. Math. Mech.* 36 (1956), 355-367. (English, French and Russian summaries)

A method of approximating the eigenvalues of

$$\frac{d^n}{dx^n} \left[ K(x) \frac{d^n y}{dx^n} \right] - (-1)^n \lambda M(x) y = 0$$

with

$$y(0) = \frac{dy}{dx}(0) = \dots = \frac{d^{n-1}y}{dx^{n-1}}(0) = 0,$$

$$\frac{d^n y}{dx^n} = \dots = \frac{d^{2n-1}y}{dx^{2n-1}}(0) = 0$$

is presented. For  $n=1$  this problem represents the torsional vibrations and for  $n=2$  the flexural vibrations of a beam of variable cross section fixed at one end.

It is easily seen that the problem can be solved explicitly when  $K$  and  $M$  are both proportional to  $e^{bx/l}$ , where  $b$  is any constant. The author treats the case where this is almost satisfied, in the sense that  $K/M$  is almost constant and  $\log KM$  is nearly linear.

The main idea is to approximate  $M$  by  $e^{b\eta}$ . Here  $\eta$  is a new independent variable, chosen so that the coefficients of  $d^{2n}y/dx^{2n}$  and of  $\lambda y$  in the resulting equation are both  $M$ . The chief problem is to find a "best" choice for the arbitrary constant  $b$ .

The author considers the transition from the approximate to the exact coefficients as a perturbation, and chooses  $b$  so that the corresponding first variation vanishes.

Several numerical examples are given, in which the computational error is small.

H. F. Weinberger.

Chien, Wei-zang. Problem of large deflection of circular plate. *Arch. Mech. Stos.* 8 (1956), 3-12. (Polish and Russian summaries)

Von Kármán equations are used to discuss the large

deflections of a uniformly loaded circular plate with various edge conditions. The central deflection, assumed to be small, is taken as the perturbation parameter. Numerical results are obtained and compared with those already known. No attempt is made to prove the convergence of the solution. The yield condition along the edge and the case of combined loading are also discussed.

B. R. Seth (Kharagpur).

Kornecki, Aleksander. A thin-walled toroidal shell under uniform pressure. *Rozprawy Inż.* 4 (1956), 119-172. (Polish. Russian and English summaries)

"In this paper is presented an approximate computation method of stress and strain in a thin-walled toroidal elastic shell of circular cross-section, limited by two parallels. The problem reduces to the determination of a complex function  $X$  satisfying the differential equation

$$(1 + \lambda \sin \alpha) \frac{d^2 X}{d\alpha^2} - \lambda \cos \alpha \frac{dX}{d\alpha} + i2k^2 \sin \alpha X = f,$$

with suitable boundary conditions... The solution of this differential equation is, in all the works known to the author, given in the form of power or trigonometric series. It is true that the homogeneous problem is solved by E. F. Zienova and V. V. Novozhilov [*Prikl. Mat. Meh.* 15 (1951), 521-530, Zienowa, Dissertation, Leningrad, 1951; MR 16, 646] by means of Bessel functions, the particular solution of the nonhomogeneous problem, however, is represented in those papers in the form of a series. This drawback of the solution of Zienova and Novozhilov is avoided in this paper, the particular solution of the nonhomogeneous problem being also expressed in a closed form by means of Bessel functions. The particular solution is based on a paper by Clark and Reissner [*Advances in Appl. Mech.*, vol. 2, McGraw-Hill, New York, 1951, pp. 93-122; MR 13, 885]. As a result, the full solution is obtained in a closed form, convenient for use, where the values of the Bessel functions can be read from the graph (which are given) or calculated by means of simple asymptotic formulae... The application of the method is illustrated by numerical examples."

(It should be noted that the results of the author are related to, but differ in the details of the illustrative examples from earlier work by R. A. Clark [*Dissertation, Mass. Inst. Tech.* 1949; *J. Math. Phys.* 29 (1950), 146-178; MR 12, 557] which had not been available to the author.)

E. Reissner (Cambridge, Mass.).

Rozovskii, M. I. Radial deformation of a hollow sphere having anisotropy and elastic after-effect. *Dokl. Akad. Nauk SSSR (N.S.)* 105 (1955), 920-923. (Russian)

The inertia term  $\rho \partial^2 u / \partial t^2$  is omitted in setting up the fundamental integro-differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{z}{r} \frac{\partial u}{\partial r} - B \frac{u}{r^2} =$$

$$\int_0^t \left[ \frac{\phi_{11}(t, \tau)}{A_{11}} \left( \frac{\partial^2 u}{\partial r^2} + \frac{z}{r} \frac{\partial u}{\partial r} \right) - \psi(t, \tau) \frac{u(\tau)}{r^2} \right] d\tau.$$

This equation is solved formally by a Neumann series.

R. C. T. Smith (Armidale).

Vekua, I. N. On a method of computing prismatic shells. *Akad. Nauk Gruz. SSR. Trudy Tbiliss. Mat. Inst. Razmadze* 21 (1955), 191-259. (Russian)

An elastic solid bounded by a cylinder  $g(x_1, x_2) = 0$  and

two surfaces  $x_3 = h^+(x_1, x_2)$ ,  $x_3 = h^-(x_1, x_2)$ , where  $2h = h^+ - h^- > 0$  is small, is called a prismatic shell. In general  $h$  is a function of  $x_1, x_2$ . A fairly rigorous treatment of such shells is given based on the idea of expanding the displacements  $u_i$ , the stresses  $x_{ij}$  and the strains  $e_{ij}$  in a series of Legendre polynomials in  $X_3$ . Thus

$$u_i \cong \sum_{n=0}^N a(n + \frac{1}{2}) u_i P_n(ax_3 - b),$$

where

$$a = \frac{2}{h^+ - h^-}, \quad b = \frac{h^+ + h^-}{h^+ - h^-},$$

$$u_i = \int_{h^-}^{h^+} u_i P_n(az - b) dz,$$

$u_i$  is a function of  $x_1, x_2$ . The relations connecting the  $u_i$  with the  $e_{ij}$  are complicated;

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \sum_{s=r}^{\infty} b_{is} u_s + \frac{1}{2} \sum_{s=r}^{\infty} b_{js} u_s \quad (i, j = 1, 2),$$

where the  $b_{ij}$  are functions of  $h^+, h^-$  and their first derivatives. The fundamental differential equations are set up by multiplying the equation of motion

$$\frac{\partial X_{ij}}{\partial x_i} + \rho X_j = \rho \frac{\partial^2 u_j}{\partial t^2}$$

by  $P_r(ax_3 - b)$  and integrating from  $h^-$  to  $h^+$ .

Existence and uniqueness theorems are proved. For  $N=0$  and  $N=1$ , various special cases are considered in detail, e.g. shells symmetric with respect to the  $Ox_1x_2$  plane and shells of revolution. *R. C. T. Smith.*

**Urban, Joachim.** *Kreisylinderschalen. Betrachtung zu zwei grundverschiedenen Schnittkraftberechnungsverfahren.* Wiss. Z. Tech. Hochsch. Dresden 5 (1955/56), 21-26.

Author discusses two methods for accounting for shear forces in shell problems. The first is based on classical methods employing assumptions based on membrane theory. The second, due to Fuchs-Steiner, is an inverse method in that the loading is determined from prescribed distortions. The two methods are applied to a shell of small curvature and calculated results are presented in detail. *H. N. Abramson* (San Antonio, Tex.).

**Vasil'ev, I. G.** *The development of the theory of elastic shells in the U.S.S.R.* Trudy Inst. Ist. Estest. Tehn. 7 (1956), 137-163. (Russian)

**Čankvetadze, G. G.** *On symmetric deformation of an elastic half-space.* Soobšč. Akad. Nauk Gruz. SSR 17 (1956), 7-14. (Russian)

The displacements are expressed in terms of a stress function  $\omega$  satisfying the biharmonic equation

$$\nabla^2 \nabla^2 \omega = 0,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2},$$

$(r, \theta, x)$  being cylindrical polar coordinates.  $\omega$  can be written as

$$\omega = \Phi_1(x, r) + x\Phi_2(x, r) + (x^2 + r^2)\Phi_3(x, r),$$

where

$$\nabla^2 \Phi_k = 0 \quad (k=1, 2, 3).$$

New variables  $\xi, \bar{\xi}$  are introduced, where  $z = H/(1-\xi)$ ,  $x = x + ir$ , and very general expressions for the  $\Phi_k$  are obtained. The particular case of a concentrated load in the  $x$ -direction acting at an interior point, the plane boundary being free, is considered in detail.

*R. C. T. Smith* (Armidale).

**★ Prager, William.** *On limiting states of deformation.* Lecture Series, no. 32. Inst. for Fluid Dynamics and Appl. Math., Univ. of Maryland, College Park, Md., 1956. 12 pp.

Materials can be classified as soft or hard depending upon whether their inelastic deformations become greater or less with increasing stress. Ductile structural materials are soft; excelsior and foam rubber are hard (under compressive stresses).

This paper is concerned with an extreme case of a hard material, defined as an ideal locking material. In simple compression an ideal locking material requires vanishingly small stress to reach a certain critical strain  $-\epsilon_0$ ; it can support any magnitude of compressive stress with the strain  $-\epsilon_0$ ; strains less than  $-\epsilon_0$  cannot be obtained by any stress. This concept represents an analogy to the ideal rigid-plastic material in which the roles of stress and strain are reversed.

Using this analogy, the author establishes a three dimensional theory for ideal locking materials, and states and proves the analogies to the theorems of limit analysis. A hollow sphere containing a concentric rigid inclusion is treated as an example. *P. G. Hodge, Jr.*

**Sen Gupta, A. M.** *Stresses in thin aeolotropic disks rotating about normal axes.* Z. Angew. Math. Mech. 35 (1955), 372-378. (German, French and Russian summaries)

Two problems of steadily rotating thin aeolotropic disks are solved in the framework of theory of plane stress. The first is that of an elliptic disk of orthotropic material whose axes of symmetry do not coincide with the major and minor axes of the ellipse. The second problem is that of a circular disk composed of isotropic material within a certain radius and of cylindrically aeolotropic material outside that radius. For both problems simple solutions in closed form are given which reduce to familiar solutions for isotropic materials.

*P. S. Symonds* (Providence, R.I.).

**Golecki, Józef.** *Boundary value problems for elastic circular rings.* Arch. Mech. Stos. 8 (1956), 123-142.

The author derives Fourier type formulae for stresses and displacements and applies them to particular problem e.g. a multilayer circular ring. *D. R. Bland* (London).

**Gaydon, F. A.; and Nuttall, H.** *The elastic-plastic bending of a circular plate by an all-round couple.* J. Mech. Phys. Solids 5 (1956), 62-65.

The complete solution is given for the title problem under the following assumptions: (1) The deformations are sufficiently small so that membrane forces may be neglected. (2) The slopes are sufficiently small so that elementary expressions for curvatures may be used. (3) The plate material is elastic-perfectly plastic with arbitrary Poisson's ratio. The Mises yield condition is used.

*P. G. Hodge, Jr.* (Brooklyn, N.Y.).

**Życzkowski, Michal.** The limit load of a thick-walled tube in a general circularly symmetrical case. *Arch. Mech. Stos.* 8 (1956), 155-178.

The theory of plastic deformation, and not plastic flow, is used to discuss the limit load of a thick walled incompressible tube. The strain and stress components are assumed to be only function of the radial distance from the axis of the tube. Simple explicit expressions are obtained for the displacements, strains and stresses. It is found that the loads can be expressed as hyperelliptic integrals. A comparison is made with the case of a solid circular cylinder, but no numerical results are given.

*B. R. Seth (Kharagpur).*

**Sokolovskii, V. V.** On forms of stable semi-arches and arches. *Prikl. Mat. Meh.* 20 (1956), 73-86. (Russian)

The paper examines the plane limiting equilibrium of a semi-arch of cohesive soil under its own weight. By a semi-arch is meant a mass of soil with a horizontal surface and with an overhang tapering to a point. The shape of the lower surface of the overhang is determined by constructing a stress field at yield in the overhang. The stress field involves a curved line of stress discontinuity. Final results are obtained numerically and in some cases are given in closed form.

*R. T. Shield.*

**Kačanov, L. M.** On the theory of plastic torsion. *Leninograd. Gos. Univ. Uč. Zap.* 135. Ser. Mat. Nauk 21 (1950), 119-126. (Russian)

Paper concerns the plastic torsion of a uniform prismatic bar. The analysis is based upon a total strain theory of plasticity in which  $G\gamma = \tau^{1+2\beta}$  ( $\gamma$ =engineering shear strain,  $\tau$ =shear stress,  $G$  and  $\beta$ =positive constants typifying mechanical behaviour of material under shear). The objective is principally to discuss cases in which the cross-sectional shape exhibits slight deviations from a circle. In particular, some results are obtained for a case when there is a local notch of special shape.

*H. G. Hopkins (Sevenoaks).*

**Ivlev, D. D.** On the use of a linear tensor connection in plasticity. *Prikl. Mat. Meh.* 20 (1956), 289-292. (Russian)

Attention being confined to homogeneous states of stress and strain, the author examines the stress-strain equations of a total strain theory of plasticity. The essential point of the analysis is to consider a procedure in which there is decomposition of the stress and strain components. The analysis is applied to the problem of a beam under combined extension and bending.

*H. G. Hopkins (Sevenoaks).*

**Cristescu, N.** Quelques observations sur le cas des déformations planes, axial symétriques, du problème dynamique de la plasticité (théorie de Prandtl-Reuss). *Com. Acad. R. P. Roum.* 6 (1956), 19-28. (Romanian. Russian and French summaries)

In the present paper the author applies the well-known Prandtl-Reuss theory (dynamical) to plane-axial symmetrical deformation of plastic bodies. Some elementary observations are made as to the nature of the two types of longitudinal waves in question.

*K. Bhagwandin.*

See also: Šestakov, p. 272; Birman, p. 316; Budiansky and Pearson, p. 331; Memoirs of the unifying study..., p. 332; Burgers, p. 348; Lance, p. 354; Hill and Power, p. 354; Glauert, p. 354; Mirels, p. 355; Ogurcov, p. 365.

## Fluid Mechanics, Acoustics

**Prem Prakash.** Harmonic analysis of the axially symmetrical incompressible viscous flow. *J. Math. Soc. Japan* 8 (1956), 102-117.

The present paper has been inspired by the method developed by the reviewer for a two dimensional flow of a viscous incompressible fluid [*Quart. Appl. Math.* 6 (1948), 1-13; *MR* 9, 631]. Considering an axially symmetric flow the author introduces in the first part the Fourier transforms  $\Psi(\omega_1, \omega_2, t)$ ,  $U(\omega_1, \omega_2, t)$ ,  $V(\omega_1, \omega_2, t)$  and  $Z(\omega_1, \omega_2, t)$  of the stream function  $\psi$  the velocity components  $u, v$  and the vorticity  $\zeta$ ; linear partial differential equations connecting  $\Psi, U, V, Z$  are given.

In the second part the kinetic energy of the fluid and its spectral decomposition  $\gamma(\omega_1, \omega_2, t)$  are considered and it is shown that  $\gamma$  has a simple expression in terms of  $\Psi$  and  $Z$ . In the third part an integro-differential equation, equivalent to the Navier-Stokes equations, is given in terms of  $U, V, Z$ . The fourth part is devoted to the case that the fluid occupies the entire space. In the fifth part a further special case, called self-superposable flow, is studied.

*J. Kampé de Fériet (Lille).*

**Czaykowski, T.** Loading conditions of tailed aircraft in longitudinal manoeuvres. *Aero. Res. Council, Rep. and Memo. no. 3001* (1955), 59 pp. (1 plate) (1956)

**Küchemann, D.** A simple method for calculating the span and chordwise loading on straight and swept wings of any given aspect ratio at subsonic speeds. *Aero. Res. Council, Rep. and Memo. no. 2935* (1952), 52 pp. (1956).

**Hay, J. A.; and Eggington, W. J.** An exact theory of a thin aerofoil with large flap deflection. *J. Roy. Aero. Soc.* 60 (1956), 753-757.

**Horlock, J. H.** An investigation of the flow in manifolds with open and closed ends. *J. Roy. Aero. Soc.* 60 (1956), 749-753.

**Krzywicki, A.** Sur le mouvement plan d'un liquide visqueux compressible. *Studia Math.* 15 (1955), 113-122.

This paper extends to compressible fluids the results of Wolibner, who showed [*Studia Math.* 12 (1951), 279-285; *MR* 13, 791] the non-existence of a steady flow of an incompressible viscous fluid in two dimensions past a fixed obstacle bounded by a closed curve  $S$ . Four conditions are imposed on admissible solutions of the equations of motion, which, at this stage of the argument, are only three equations for the four unknowns ( $u, v, p, \rho$ ). Three of these conditions concern the boundary conditions on  $S$ , the finiteness of kinetic energy, and the continuity of the unknowns and their derivatives; the fourth demands the existence of a sequence of positive numbers  $a_n$  (tending to infinity with  $n$ ) and a positive number  $M$ , such that, in polar coordinates,

$$(*) \quad \int_0^{2\pi} d\phi \int_{a_n}^{8a_n} \rho r dr \leq M a_n^2, \quad \int_0^{2\pi} d\phi \int_{a_n}^{8a_n} \rho^{-1} r dr \leq M a_n^2.$$

Formulae, involving  $n \rightarrow \infty$ , are obtained for the components of the total force acting on  $S$ , and from these formulae it results that the force is zero if the motion is steady. Finally, the conditions (\*) are withdrawn, and the usual adiabatic ( $p, \rho$ ) relation is imposed, together with conditions of boundedness on  $p$ , on radial velocity,

and on expansion; then no steady motion exists. It is stated that a slight modification of the argument establishes the non-existence of steady motions for a heterogeneous incompressible viscous fluid. *J. L. Synge.*

**Krzywicki, A.** Sur le mouvement plan du liquide visqueux compressible. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 185-187.

A résumé of the paper reviewed above. *J. L. Synge.*

**Krzywicki, A.** Sur la force latérale exercée sur un obstacle par un liquide visqueux et compressible. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 241-242.

This paper extends to the 3-dimensional motion of a compressible viscous fluid the theory developed in the papers reviewed above. A condition analogous to (\*) is imposed, and formulae obtained for the total force on the obstacle. The lateral force vanishes in the case of steady motion. No  $(\phi, \rho)$  relation is imposed. *J. L. Synge.*

**Krzywicki, A.** Sur la force latérale exercée sur un obstacle par un liquide visqueux compressible. Studia Math. 15 (1956), 174-181.

The author gives details of theory reported in the paper reviewed above; he discussed the corresponding problem in plane motion in the paper reviewed third above. A closed rigid surface  $\Sigma$  moves with constant velocity  $(0, 0, W)$  through a compressible fluid with two viscosity coefficients  $\mu, \nu$ . The coordinates  $(x, y, z)$  are tied to  $\Sigma$ , but the velocity  $(u, v, w)$  is referred to axes fixed in space. The assumptions are simple: (i) the fluid sticks to  $\Sigma$ , (ii) the total kinetic energy is finite, (iii) the mean density and mean reciprocal density are finite, (iv) continuity of variables and their derivatives. It is remarkable that no  $(\phi, \rho)$  relationship is needed. The plan is to enclose  $\Sigma$  within an infinite sequence of cylindrical surfaces  $\Phi$  (length  $2h$ , radius  $R$ ). From the equations of motion and of continuity it follows that the component  $P_z$  of lateral force on  $\Sigma$  may be expressed in the form

$$(*) \quad P_z = - \int_{\Omega} \frac{\partial}{\partial t} (\rho u) d\omega + \int_{\Phi} \rho u \{ u n_x + v n_y + (w - W) n_z \} d\sigma \\ + \int_{\Phi} \phi n_x d\sigma - \nu \int_{\Phi} \theta n_z d\sigma - \mu \int_{\Phi} \frac{\partial u}{\partial n} d\sigma,$$

where  $\Omega$  is the volume between  $\Sigma$  and  $\Phi$ . Since there is no assumption bounding  $\phi$ , it is necessary to change  $\phi$  into a partial derivative in order to use the equations of motion; this is done by the formula

$$\int_{\Gamma} \phi \cos \phi d\sigma = - \int_{\Gamma} \frac{\partial \phi}{\partial \phi} \sin \phi d\sigma,$$

with integration over the curved part ( $\Gamma$ ) of  $\Phi$ ,  $\phi$  being an azimuthal angle. In this way (\*) is reduced to the form

$$P_z + \int_{\Omega} \frac{\partial}{\partial t} (\rho u) d\omega + R \int_{\Gamma} \frac{\partial}{\partial t} (\rho u \phi) \sin \phi d\sigma = F(R, h)$$

where  $F(R, h)$  is a sum of integrals taken over  $\Gamma$  and the ends of  $\Phi$ , the integrands being independent of  $\phi$  and involving the density and velocity components, with their first and second derivatives. The next step is to prove, using the inequality of Schwartz, that

$$\lim_{\gamma \rightarrow \infty} \frac{1}{\gamma^2 \eta^2} \int_{\gamma}^{2\gamma} d\beta \int_{\beta}^{2\beta} d\alpha \int_{\gamma}^{2\gamma} dR \int_{\eta}^{2\eta} d\zeta \int_{\zeta}^{2\zeta} F(R, h) dh = 0 \\ (\gamma^2 \leq \eta \leq 2\gamma^2).$$

By application of such integration processes, there follows the existence of sequences  $\{R_n\}$  and  $\{h_n\}$  such that

$$P = - \lim_{n \rightarrow \infty} \left[ \int_{\Omega_n} \frac{\partial}{\partial t} (\rho u) d\omega + R_n \int_{\Gamma_n} \frac{\partial}{\partial t} (\rho u) \sin \phi d\sigma \right],$$

and of course a similar expression for  $P_y$ , the other component of lateral force. In the particular case of steady motion, the lateral force is zero. *J. L. Synge.*

**Krzywicki, A.** Sur la force frontale exercée sur un obstacle par un liquide visqueux compressible. Studia Math. 15 (1956), 252-266.

The physical problem is as in the paper reviewed above, but now attention is concentrated chiefly on the force  $P_z$  in the direction of the motion of  $\Sigma$ . The general technique is the same, viz. the formation of an infinite sequence of cylinders containing  $\Sigma$ , but the basic assumptions are changed. Using a variety of assumptions, the author obtains six different expressions for  $P_z$ . He regards some of these assumptions as not quite natural, and for brevity this review will deal only with Theorems A and B, for which the assumptions are simple. For A the assumptions are (a) the fluid sticks to  $\Sigma$ , (b)  $\int \rho(|u|^{3/2} + |v|^{3/2} + |w|^{3/2}) d\omega$  is finite for integration through the whole fluid, (c)  $u, v, w$  are bounded and (d) a constant  $M$  exists such that  $\int \rho d\omega \leq M \int d\omega$ ,  $\int \rho^{-1} d\omega \leq M \int d\omega$ , the integrals being taken through the part exterior to  $\Sigma$  of any arbitrary circular cylinder with the  $x$ -axis for axis, symmetric with respect to the plane  $yz$ , with height equal to radius. Theorem A states that there exist two sequences of cylindrical surfaces  $\psi_n, \psi'_n$ , increasing indefinitely with  $\frac{1}{2} \leq R_n/h_n, R'_n/h'_n \leq 2$ , such that

$$P_y = - \lim_{n \rightarrow \infty} \left[ \int_{\Omega_n} \frac{\partial}{\partial t} (\rho v) d\omega - R_n \int_{\Gamma_n} \frac{\partial}{\partial t} (\rho u) \cos \phi d\sigma \right],$$

$$P_z = - \lim_{n \rightarrow \infty} \left[ \int_{\Omega_n} \frac{\partial}{\partial t} (\rho w) d\omega + R'_n \int_{\Gamma'_n} \frac{\partial}{\partial t} (\rho u) \sin \phi d\sigma \right],$$

where  $\Omega_n$  is the volume between  $\Sigma$  and  $\psi_n$ ,  $\Gamma_n$  the lateral surface of  $\psi_n$ ,  $R_n$  its radius and  $h_n$  its height. The author states that the proof is analogous to that of the theorem in the paper reviewed above, the Hölder inequality replacing the Schwarz inequality. In B the assumptions of A are augmented by the boundedness of  $\phi$  and  $\theta$  (the expansion of the fluid) and by a characteristic equation of the form  $\phi/\rho^\lambda = \text{const}$  ( $0 \leq \lambda < 1$ ), so that the case of incompressibility is included. Theorem B expresses  $P_z$  as the limit of the sum of four integrals. This is more complicated than the formula given by A, and differs from it in an important respect: it contains a positive-definite integral which vanishes only if  $(u, v, w)$  are constants. Comparison of the two expressions leads to the conclusion that, under the conditions of B, no steady motion exists. For if it did, then A would imply  $P_z = 0$ , and (through the aforesaid positive-definite integral)  $u = v = 0, w = W$ ; hence  $\phi = \text{const}, \rho = \text{const}$ , and the condition (b) would be violated. *J. L. Synge (Dublin).*

**Power, G.; and Scott-Hutton, D. L.** The slow shearing motion of a liquid past a semi-infinite plane. Pacific J. Math. 6 (1956), 327-349.

This paper deals with the problem of slow two-dimensional flow of a viscous incompressible fluid bounded by two parallel planes which are, respectively, infinite and semi-infinite. The motion at great distances from the planes is a simple shear and between the planes it approaches for downstream a flow due to a constant

pressure gradient. The solution obtained is biharmonic and contains infinite number of arbitrary constants which cannot be chosen to satisfy the nonslip conditions at the walls.

Y. H. Kuo (Peking).

**Oldroyd, J. G.; and Thomas, R. H.** The motion of a cylinder in rotating liquid with general elastic and viscous properties. *Quart. J. Mech. Appl. Math.* 9 (1956), 136-139.

Es wird gezeigt, dass jede zweidimensionale Bewegung einer vollkommen inkompressiblen Flüssigkeit, die einem beliebigen Schubspannungsgesetz folgt, unverändert bleibt, wenn das ganze System um eine Achse senkrecht zur Bewegungsebene mit konstanter Winkelgeschwindigkeit rotiert. Vorausgesetzt ist, dass die Relativbewegung zwischen Flüssigkeit und Störkörper unverändert bleibt. Die bei konstanter Winkelgeschwindigkeit  $\omega$  des Systems auf den Körper ausgeübte Kraft ist

$$F_2 = F_1 + M' [2\omega \Lambda \dot{R} + \omega \Lambda (\omega \Lambda R)],$$

wobei  $F_1$  die auf den Körper bei ruhendem System wirkende Kraft,  $M'$  die Masse der verdrängten Flüssigkeit und  $R$  der die Lage des Körpers relativ zum drehenden System beschreibende Vektor ist.

L. Speidel.

**Lance, G. N.** Motion of a viscous fluid in a tube which is subjected to a series of pulses. *Quart. Appl. Math.* 14 (1956), 312-315.

An analysis is made of the flow of a viscous incompressible fluid in a circular pipe or two-dimensional channel subjected to a series of pulses parallel to the flow direction. The results show that when the pulses act in the opposite direction to the pressure gradient the total flow can be stopped under certain conditions. D. W. Dunn.

**Hill, R.; and Power, G.** Extremum principles for slow viscous flow and the approximate calculation of drag. *Quart. J. Mech. Appl. Math.* 9 (1956), 313-319.

The authors first draw attention to the analogy between an incompressible elastic solid and a Newtonian viscous fluid in quasi-static flow. They next consider the particular case of a rigid body moving in a Newtonian fluid contained within fixed finite or infinite boundaries. They show that the analogy to the principles of minimum potential and complementary energies in elasticity lead to upper and lower bounds on rate of dissipation of energy in the fluid. In particular, if the rigid body undergoes no rotation, bounds on the drag are obtained.

The following interesting corollaries to the extremum principles are then established: (a) The drag on a body  $S$  is equal or greater than the drag on any body wholly contained in  $S$ , other conditions being equal. (b) The drag on a given body in a container  $C$  is equal or less than the drag on the same body in any container completely contained in  $C$ . (c) The drag on a body  $S$  is not decreased by the presence of other fixed or free bodies in the fluid. (d) The apparent viscosity of a fluid containing a solid suspension is a monotonically increasing function of the volume concentration of solute.

The paper closes with three examples of approximate solutions, two of which are compared with exact solutions available in the literature. In the final example, close upper and lower bounds are obtained to a previously untreated problem.

P. G. Hodge, Jr.

**Glauert, M. B.** The laminar boundary layer on oscillating plates and cylinders. *J. Fluid Mech.* 1 (1956), 97-110.

Author studies the two-dimensional laminar boundary

layer on an infinite flat plate normal to a stream, the plate making transverse oscillations in its own plane. The exact solution (which also satisfies the full Navier-Stokes equations) depends on a single ordinary differential equation, containing the frequency of oscillation as a parameter. Series methods are employed to evaluate the solution for small and large values of the frequency, and numerical values of the skin friction are obtained over the whole frequency range.

The results for a flat plate are used to describe the boundary layer in the neighbourhood of the front stagnation point on a cylinder of arbitrary section making transverse or rotational oscillations. An estimate is made of the fluctuating torque on a circular cylinder making transverse oscillations.

A. E. Green.

**Smith, A. M. O.** Rapid laminar boundary-layer calculations by piecewise application of similar solutions. *J. Aero. Sci.* 23 (1956), 901-912.

A method is presented for the rapid calculation of the incompressible laminar boundary layer around a two-dimensional or rotationally-symmetrical body. The Kármán momentum integral equation is not involved, and use is made of a coarse step-by-step procedure in which each segment of the velocity distribution is approximated by one of the Falkner-Skan family of similar flows. Generally, only four steps are needed between the forward stagnation point and the pressure peak. Accuracy is at least as good as in any other one-parameter approximate method. In contrast to the Kármán-Pohlhausen procedure, which sometimes fails to predict separation that actually exists, the present method predicts separation somewhat early. D. W. Dunn (Baltimore, Md.).

**Monin, A. S.** The equation of turbulent diffusion. *Dokl. Akad. Nauk SSSR (N.S.)* 105 (1955), 256-259. (Russian)

La diffusion turbulente est causée par l'action des tourbillons des différentes dimensions dans un milieu turbulent. La distance entre deux particules qui diffusent change d'une façon sensible seulement sous l'action des tourbillons dont les dimensions sont de même ordre que la distance entre les particules. En moyenne la variation de la distance est très faible tant que la distance est petite et devient très grande lorsque la distance augmente sensiblement, c'est-à-dire que le coefficient de diffusion  $K$  est fonction de l'échelle du phénomène.

Les expériences ont confirmé la règle  $K \sim l^{4/3}$  dont l'explication a été fournie par Kolmogoroff [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 30 (1941), 301-305; MR 2, 327] et Obuhov [Izv. Akad. Nauk SSSR. Ser. Geograf. Geofiz. 1941, 453-466; MR 4, 121]. L'auteur se propose aussi de former l'équation de la concentration du mélange qui diffuse ( $q$ ).

Malheureusement la méthode bien connue de Richardson est peu efficace pour le calcul de  $q$ . En introduisant quelques hypothèses simples l'auteur arrive à trouver l'expression de  $q$ . Pour une source non stationnaire on a

$$q_1(r, t) = \frac{1}{\pi \varepsilon_1^{1/3} r^{4/3}} f_1\left(\frac{r^2}{\varepsilon_1 t^3}\right),$$

ou  $r^2 = x^2 + y^2$  et  $\varepsilon_1$  est proportionnelle à la vitesse de dissipation de l'énergie turbulente

$$f_1(\xi) = \frac{1}{2^{2/3}} \frac{\Gamma(2/3)}{\Gamma(1/3)} - \frac{\xi^{1/6}}{2\sqrt{3}} {}^{12/27}\xi W_{(-1/20)(1/3)}\left(\frac{4}{27\xi}\right),$$

où  $W$  est la fonction hypergéométrique dégénérée de Whittaker.

Pour  $t$  très grand on a pour une distribution stationnaire:

$$q_2(r) = \frac{1}{2^{2/3}\pi} \frac{\Gamma(2/3)}{\Gamma(1/3)} \varepsilon_1^{-1/3} j^{-4/3}.$$

M. Kiveliovitch (Paris).

**Monin, A. S. Horizontal intermingling in the atmosphere.** Izv. Akad. Nauk SSSR. Ser. Geofiz. 1956, 327-345. (Russian)

L'auteur reprend sous une forme plus complète et plus détaillée ses recherches discutées dans le travail analysé ci-dessus.

En utilisant les résultats obtenus il les applique au cas de la diffusion horizontale dans un milieu en mouvement par rapport à une source ponctuelle. M. Kiveliovitch.

**Matschinski, M. Sur les méthodes d'élimination des fluctuations aléatoires du champ hydrodynamique.** J. Sci. Météorol. 7 (1955), 339-357. (Spanish and English summaries)

Les méthodes habituellement employées pour traiter les problèmes d'hydrodynamique turbulente introduisent des concepts de moyennes dont l'inconvénient réside dans ce fait qu'on obtient toujours moins d'équations que d'inconnues. L'auteur propose une "méthode d'élimination des fluctuations" dans laquelle  $1^0$  — il considère comme "assez faible" la partie fluctuante de la vitesse  $2^0$  — il suppose que la divergence des fluctuations de vitesse est nulle, mais leur "circulation" ou leur rotationnel ne l'est pas. Ces hypothèses permettent, dans le cas des écoulements à deux dimensions, d'écrire un système de deux équations pour déterminer la vitesse macroscopique, puis une équation pour déterminer la "pression microscopique". Pour les écoulements à trois dimensions, il faut ajouter des hypothèses sur les grandeurs relatives des composantes des fluctuations de vitesse (fluctuations sphériques ou filiformes). Des considérations de similitude, jointes à une hypothèse sur la structure physique des fluctuations tourbillonnaires, permettent enfin de relier la pression microscopique à la pression macroscopique. J. Bass (Paris).

**Coles, Donald. The law of the wake in the turbulent boundary layer.** J. Fluid Mech. 1 (1956), 191-226.

The results of this paper are based on an extensive, but careful, survey of the experimental measurements of mean velocity profiles in various two-dimensional incompressible boundary layer flows. It is concluded that the mean-velocity profile can be represented by a linear combination of two universal functions. The first of these is the well-known logarithmic law (called the law of the wall). The second (called the law of the wake) is characterized by the profile at a point at which the wall shearing stress is zero. The latter function is tabulated. A physical interpretation of the wake function is given. Finally the possibility of using these two universal functions in representing three-dimensional flows is discussed.

R. C. DiPrima (Culver City, Calif.).

**Acharya, Y. V. G. Spectrum of axi-symmetric turbulence in a stream.** Proc. Indian Acad. Sci. Sect. A. 44 (1956), 63-71.

The passage of axi-symmetric turbulence in a suddenly contracting stream has been investigated. A form of axi-

symmetric tensor suitable for this study has been derived. Assuming that the form of turbulence is initially axi-symmetric, upstream of the contraction, the spectral forms downstream have been given. The ratio of the turbulence levels, on either side of the contraction, has been evaluated in integral form. Further, the integrals have been solved, assuming that the defining scalars are functions only of the wave number  $\kappa$ . The method of evaluating one-dimensional spectra has been indicated. The whole investigation is based on the linearised approach. Author's summary.

**Torre, C.; und Martinek, J. Zur Berechnung der turbulenten Strömung.** Z. Angew. Math. Mech. Sonderheft (1956), S46-S48.

**Walz, A. Näherungstheorie für kompressible turbulente Grenzschichten.** Z. Angew. Math. Mech. Sonderheft (1956), S40-S56.

**Pack, D. C. The oscillations of a supersonic gas jet embedded in a supersonic stream.** J. Aero. Sci. 23 (1956), 747-753, 764.

On the basis of linearized theory, the author proposes to study the general behavior of a supersonic jet emerging into a medium in a supersonic stream. In the case of plane jet which was previously investigated by Pai [J. Aero. Sci. 19 (1952), 61-65] and Kawamura [J. Phys. Soc. Japan 7 (1952), 482-485; MR 14, 425], the new findings confirm those of the earlier studies. In the case of axially symmetric jet, the results show that the jet boundary has singularities in boundary gradient which are alternatively logarithmic singularities and simple discontinuities and that the boundary approaches very rapidly the asymptotic thickness of the jet. Y. H. Kuo (Peking).

**Mirels, Harold. Attenuation in a shock tube due to unsteady-boundary-layer action.** NACA Tech. Note no. 3278 (1956), 1+60 pp.

"A method is presented for obtaining the attenuation of a shock wave in a shock tube due to the unsteady boundary layer along the shock-tube walls. It is assumed that the boundary layer is thin relative to the tube diameter and induces one-dimensional longitudinal pressure waves whose strength is proportional to the vertical velocity at the edge of the boundary layer. The contributions of the various regions in a shock tube to shock attenuation are indicated. The method is shown to be in reasonably good agreement with existing experimental data." (Author's summary.) R. Finn.

**McCrea, W. H. Shock waves in steady radial motion under gravity.** Astrophys. J. 124 (1956), 461-468.

The investigation is an elaboration of some work by Bondi [Monthly Not. Roy. Astr. Soc. 112 (1952), 195-204; MR 14, 212]. A star of constant mass is surrounded by a spherically symmetric cloud of gas extending to infinity, where its density and pressure are uniform, and its velocity is zero. The gas falls into the star and a steady state of motion is assumed. It is shown that a standing shock wave can occur in the gas; outside such a shock (the supersonic side) the solution for the flow was found by Bondi; inside the shock (the subsonic side) it is shown that the motion can be represented by one of a set of solutions found by Bondi, but not interpreted by him. A discussion is given of the way in which a steady state of this kind might be set up and it is also suggested that it

throws light on accretion by binary systems. A numerical illustration is provided. *G. C. McVittie* (Urbana, Ill.).

**Garner, H. C.** Multhopp's subsonic lifting-surface theory of wings in slow pitching oscillations. *Aero. Res. Council. Rep. and Memo. no. 2885* (1952), 49 pp. (1956).

**Lighthill, M. J.** The wave drag at zero lift of slender delta wings and similar configurations. *J. Fluid Mech.* 1 (1956), 337-348.

This is an application of the theory of Ward [*Quart. J. Mech. Appl. Math.* 2 (1949), 75-97; MR 10, 644] to bodies terminating in trailing edges at right angles to the stream. The author notes that Ward excluded bodies whose cross-sections have large curvature, but makes the conjecture that the theory can still be applied if there is no flow of fluid around such points of high curvature. He also remarks that Ward's theory has often been misinterpreted as stating that the drag of a slender body is always that of an equivalent body of revolution. In general there is an additional term dependent upon the shape of the rear cross section. This term is evaluated here for the class of bodies mentioned above. Resulting drag values are compared with those calculated for delta wings by Puckett [*J. Aero. Sci.* 13 (1946), 475-484; MR 8, 109] and Squire [*Aero. Res. Council, Rep. and Memo. no. 2549* (1951); MR 14, 109] by linearized wing theory, and with experimental results. Other applications are also made.

*W. R. Sears* (Ithaca, N.Y.).

**Jones, W. P.** The oscillating aerofoil in subsonic flow. *Aero. Res. Council, Rep. and Memo. no. 2921* (1953), 16 pp. (1956).

**Valdenazzi, L.** On the form of a jet issuing from a swirl atomizer. *Ing.-Arch.* 24 (1956), 330-340.

See also: Hesse, p. 348; Oswald, p. 348; Sedney, p. 356; Chandrasekhar, p. 357; Chambré, p. 358; Bhattacharya, p. 362; Adem, p. 365; Opatowski and Schmidt, p. 367.

### Optics, Electromagnetic Theory, Circuits

**Picht, Johannes.** Über aplanatisch abbildende Flächen. *Wiss. Z. Pädagog. Hochsch. Potsdam* 1 (1954/1955), 95-99.

The author calculates the differential equation for a system consisting of a single surface, such that spherical aberration and/or sine condition are corrected. He obtains the well-known result that both conditions can only be fulfilled if the refracting surface is a sphere.

*M. Herzberger* (Rochester, N.Y.).

**Sedney, R.** Geometrical optics of angular stratified media. *Quart. Appl. Math.* 14 (1956), 225-230.

An angular stratified (optical) medium is one in which the index of refraction is homogeneous of degree zero and rotationally symmetric, i.e.  $n=n[(x^2+y^2)^{1/2}/z]$ . Some properties of the paths of light rays traversing an angular stratified cone are established. Further, two conjectures are reported to have been verified by numerical integration, in special cases, of the path equations. An application to the interferometric method of observing super-

sonic flows shows that a conical flow test is valid including the refraction effect. *G. L. Walker.*

**Linfoot, E. H.** Transmission factors and optical design. *J. Opt. Soc. Amer.* 46 (1956), 740-752.

Criteria for the evaluation of the quality of an optical image are discussed. Some shortcomings of earlier methods of image assessment are noted and new criteria are proposed. Monochromatic as well as polychromatic images are considered. *E. Wolf* (New York, N.Y.).

**Steel, W. H.** The defocused image of sinusoidal gratings. *Opt. Acta* 3 (1956), 65-74.

The paper is concerned with the light distribution in out-of-focus images of a sinusoidal grating formed by an aberration-free system with a circular or annular aperture. Coherent as well as partially coherent illumination is considered.

The transmission factor (frequency response function) is evaluated in terms of the incomplete Bessel and Struve functions and the results are displayed graphically and also in tabulated form. The effect of limiting the grating to a finite number of periods is also considered and the results are applied to evaluation of resolving power tests.

*E. Wolf* (New York, N.Y.).

**Durand, Emile.** Les fonctions discontinues de l'électrostatique et de la magnétostatique. *Ann. Fac. Sci. Univ. Toulouse* (4) 19 (1955), 161-174 (1956).

The general formulas derived treat the discontinuity of a surface integral involving the gradient of  $1/r$  and the discontinuity of a line integral (along a curve) involving the gradient of  $\log r$ . These general formulas are applied to discontinuities in electrostatics and magnetostatics. Surface distributions of charges and dipoles and volume distributions of charges are considered. It is indicated that this treatment is equally applicable to electric charges and to magnetic masses. Discontinuities in problems involving surface and volume distributions of currents are also studied.

*J. E. Rosenthal* (Passaic, N.J.).

**Teisseyre, R.** New method of solving the diffraction problem for a dipole field. *Bull. Acad. Polon. Sci. Cl. III.* 4 (1956), 433-438.

The author simplifies his earlier discussion of the diffraction of a perfectly conducting wedge by a dipole [same *Bull.* 3 (1955), 157-162, 523-526; MR 16, 1075; 17, 807].

*A. E. Heins* (Pittsburgh, Pa.).

**Halfin, L. A.** Remark on the method of introduction of interaction with the external field. *Vestnik Leningrad. Univ.* 11 (1956), no. 10, 39-42. (Russian)

The author considers a classical particle interacting with an external electromagnetic field. He assumes that the interaction Lagrangian 1) depends only on the electromagnetic field through the vector and scalar potential expressed at the position of the electron, 2) is gauge invariant, and 3) does not give rise to an increase in the order of the time-derivatives occurring in the equation of motion. He proves that the interaction Lagrangian is a constant multiple of the customary one.

*A. S. Wightman* (Princeton, N.J.).

**Cohn, Harvey.** Stability configurations of electrons on a sphere. *Math. Tables Aids Comput.* 10 (1956), 117-120.

The author considers the problem of determining a

stable configuration of  $n$  electrons bound to the unit sphere and interacting under mutual Newtonian repulsion. He begins by quoting the results of Föppl [J. Reine Angew. Math. 141 (1912), 251-302] for some small values of  $n$ , namely

$1+m+1$  ( $m=3, 4, 5$ );  $1+m+m+1$  ( $m=3, 4, 5, 6$ ). (This notation is in terms of parallel rings: " $1+m+m+1$ " means one electron at each pole and  $m$  on each of two parallels of latitude, so that these  $2m$  are at the vertices of an  $m$ -gonal antiprism. Actually the author's description is valid only when  $m$  is odd.) He has used the IBM 701 to compute the first unknown case:  $n=9$ . The conclusion is  $3+3+3$ , i.e., the vertices of a tall triangular prism along with the central projections of the centres of its three rectangular faces. The parameter implied in this description is computed to six significant figures [cf. L. L. Whyte, Amer. Math. Monthly 59, 606-611 (1952); MR 14, 310]. For  $n=11$  he finds an arrangement of the form  $3+5+3$ , where the 3's indicate isosceles triangles and the 5 an irregular pentagon on the equator.

H. S. M. Coxeter (Toronto, Ont.).

Ferla, Ambrogio. Sulla propagazione delle onde elettromagnetiche nei corpi omogenei in moto. Boll. Un. Mat. Ital. (3) 11 (1956), 229-237.

L'Autore ritrova con un metodo vettoriale che si distingue per la sua semplicità l'equazione risolvente dei fenomeni elettromagnetici nei mezzi conduttori in moto traslatorio rettilineo uniforme, stabilita da G. Carini [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 19(88) (1955), 152-158; MR 17, 561] seguendo i criteri di G. Lampariello [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 17 (1954), 222-228; MR 17, 216].

Un confronto è istituito fra codesto metodo e quello esposto da T. Zeuli per raggiungere lo stesso scopo [veda il lavoro recensito sotto]. G. Lampariello (Roma).

Zeuli, Tino. Sui fenomeni elettromagnetici nei corpi omogenei elettricamente conduttori, in moto traslatorio uniforme. Univ. e Politec. Torino. Rend. Sem. Mat. 14 (1954-55), 141-158.

Il principale oggetto di questo lavoro è di stabilire un'equazione risolvente dei fenomeni elettromagnetici nei conduttori in moto (traslatorio rettilineo uniforme rispetto ad un qualunque sistema inerziale), più generale di quella di G. Lampariello per i mezzi dielettrici.

Il risultato era già stato pubblicato da G. Carini [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 19(88) (1955), 152-158; MR 17, 561]. G. Lampariello.

Zeuli, Tino. Alcune considerazioni sulle equazioni della elettrodinamica nei corpi in moto traslatorio uniforme. Boll. Un. Mat. Ital. (3) 11 (1956), 189-197.

L'Autore completa alcuni risultati stabiliti nel lavoro recensito sopra, ricercando l'equazione della densità spaziale di carica. G. Lampariello (Roma).

Konorski, Boleslaw. Verallgemeinerung des Coulombschen Grundgesetzes. Arch. Elektrotech. 42 (1956), 381-397.

\* Vainshtein, L. A. Surface electromagnetic waves over corrugated structures. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 17 pp.

Translated from Ž. Teh. Fiz. 26 (1956), no. 2, 385-397.

\* Troitskii, V. N. Ultra-short wave reflection from laminar inhomogeneities of the troposphere. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 14 pp.

Translated from Radiotekhnika 11, no. 1, 1956, 7-16.

Hoehnke, Hans-Jürgen. Die Konstanten der Wellenleitungen. Eine Ausdehnung der Abrahamschen Leitungstheorie. Arch. Elektrotech. 42 (1956), 426-448.

Papadopoulos, V. M. Scattering by a semi-infinite resistive strip of dominant-mode propagation in an infinite rectangular wave-guide. Proc. Cambridge Philos. Soc. 52 (1956), 553-563.

The scattering of the dominant transverse electric mode in an infinitely perfectly conducting rectangular guide by a semi-infinite strip, centrally placed and parallel to the electric field, is calculated by the use of Laplace transforms; (Reviewer's comment: Essentially by an application of the factoring method due to Wiener and Hopf). Formulae are obtained for the amplitude of the scattered waves and numerical examples are presented.

A. E. Heins (Pittsburgh, Pa.).

Caprioli, Luigi. Onde e. m. di tipo trasversale nelle guide d'onda rettilinee e con dielettrico eterogeneo. Boll. Un. Mat. Ital. (3) 11 (1956), 200-202.

Further comments on an earlier paper [Atti 40 Congresso Un. Mat. Ital., Taormina, 1951, v. 2, Edizioni Cremonese, 1953, pp. 478-483; MR 15, 184] as a result of a discussion with F. Sbrana.

A. E. Heins.

De Socio, Marialuisa. Sulle condizioni al contorno per le guide imperfettamente conduttrici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 469-476.

In place of the complicated boundary conditions needed for a rigorous treatment of the problem of a wave guide with finite conducting walls, S. A. Schelkunoff [Electromagnetic waves, Van Nostrand, New York, 1943, p. 320] has suggested the condition

$$\mathbf{E}_t = (\mu\omega/2\gamma)^{1/2} (1+i) \mathbf{H}_t \times \mathbf{n}$$

( $\gamma$  conductivity,  $\mu$  permeability,  $\omega$  frequency,  $\mathbf{n}$  unit exterior normal to wall) relating the tangential field components, and P. Baudoux [Rev. H. F. Belg. 3 (1955), 17-26] has proposed the condition  $\mathbf{H}_t = \gamma_s \mathbf{n} \times \mathbf{E}_t$  ( $\gamma_s$  surface conductivity). The author determines the propagation modes in two cases in which resort to these formulas is unnecessary: (1) in the space bounded by parallel planes between two conducting half-spaces; and (2) in the infinitely long space bounded by a conducting circular-cylindrical shell of infinite thickness. She studies the conditions in both cases under which each of the above formulas gives a valid approximation to the solution.

R. N. Goss (San Diego, Calif.).

Chandrasekhar, S. On the stability of the simplest solution of the equations of hydromagnetics. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 273-276.

This paper deals with a class of exact stationary solutions of the equations of hydromagnetics, namely

$$v_i = \pm H_i \left( \frac{\mu}{4\pi\rho} \right)^{1/2} \text{ and } p + \rho V + \frac{\mu |H|^2}{8\pi} = \text{const},$$

where  $H_i$  and  $v_i$  ( $i=1, 2, 3$ ) denote the components of the

magnetic intensity and of the velocity respectively;  $V$  the gravitational potential;  $p$ , the pressure;  $\rho$ , the density; and  $\mu$  the coefficient of magnetic permeability. It is shown that the solution is neutrally stable and a variational method is given for the determination of the characteristic frequency. Special examples and further details are forthcoming in other papers to be published.

C. C. Lin (Cambridge, Mass.).

- ★ Cherry, E. Colin. **Duality, partial duality and contact-transformations.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 323-347. Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1956.

This paper is concerned with the application of Hamiltonian methods to the theory of loss-free electrical networks and the interpretation of contact transformations in terms of the transformation of networks. Details are given only for linear circuits, but the applications of the method to nonlinear systems is recommended.

J. L. Synge (Dublin).

- Ekberg, Stellan. **Mathematical analysis of coaxial cables.** Kungl. Tekn. Högsk. Handl. Stockholm no. 107 (1956), 116 pp.

- ★ Bystrov, G. N. **To the question of the dependence between the observability of an object and the number of 'illuminating' impulses.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.  
Translated from Radiotekhnika 11, no. 2, 1956, pp. 74-76.

- ★ Povarov, G. N. **On a method of analyzing symmetric switching circuits.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 4 pp.  
Translated from Avtomatika i Telemekhanika 16 (1955), no. 4, 364-365.

See also: Steel and Ward, p. 300; Arsac, p. 303; Memoirs of the unifying study. . . , p. 332; Keitel, p. 338; Toupin, p. 349; Pease and Pease, p. 360; Hack, p. 360; Durand, p. 362; Vescan, p. 362.

### Thermodynamics and Heat

- ★ Temkin, A. G. **Theorem on the maximum temperature gradient.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 11 pp.  
Translated from Ž. Teh. Fiz. 25 (1955), no. 3, 534-540.

- Baratta, Maria Antonietta. **Sopra un problema non lineare di propagazione del calore in un mezzo dotato di simmetria sferica.** Boll. Un. Mat. Ital. (3) 11 (1956), 427-431.

Let  $r$  represent the distance from the origin  $(0, 0, 0)$  of coordinates to the point  $(x, y, z)$ . Let  $S_1$  and  $S_2$  denote respectively the domains  $0 \leq r < a$ , and  $a < r < b$ . The functions  $U^{(i)}(r, t)$ ,  $i=1, 2$  give the temperature at time  $t$  in the mediums  $S_i$ . Assuming that the surface  $r=a$  is a source  $T=T(t)$  of heat and that the film transfer factor of this surface is a function of  $\Phi=U^{(2)}(a+, t)-U^{(1)}(a-, t)$ , the author is led to the heat conduction boundary value

problem:

$$U_{rr}^{(i)} + 2r^{-1}U_r^{(i)} = U_t^{(i)}, \text{ in } S_i, t > 0, i=1, 2,$$

$$k_i U_r^{(i)} = (-1)^i (G(\Phi) - C_i T(t)), r=a, t > 0,$$

$$U^{(i)}=0, \text{ in } S_i \text{ for } t=0; k_i U_r^{(i)} = -F, \text{ for } r=b, t > 0.$$

Here  $k_i, C_i, F$  are constants, and the functions  $T(t), G(\Phi)$  are assumed to be known functions of their arguments. The object of the present paper is to show that the question of solving the above heat transfer problem can be reduced to the question of solving a nonlinear Volterra integral equation of the second kind in which  $\Phi$  is the unknown function [for references to similar problems see M. A. Baratta, Riv. Mat. Univ. Parma 5 (1954), 363-371; MR 17, 271]. F. G. Dressel (Durham, N.C.).

- Chambré, P. L. **On the dynamics of phase growth.** Quart. J. Mech. Appl. Math. 9 (1956), 224-233.

A form of Stefan's problem is treated here. Let a body of supercooled fluid, of infinite extent, solidify in such a way that the solidification front is a plane or a circular cylindrical surface or a spherical surface. The rate of advance of the front depends on the rate at which latent heat of solidification is transmitted away from the front. The temperature of the solid is assumed to maintain a constant and uniform value  $T_s$ . In contrast to assumptions made in forms of Stefan's problem treated by other authors, the densities of the fluid and solid are assumed here to be unequal. Thus convection of the fluid with a velocity  $u(r, t)$  normal to the front is involved, where the coordinate  $r$  denotes distance in that normal direction. The author writes a nonlinear system of partial differential equations that is to be satisfied by the velocity  $u(r, t)$ , the temperatures  $T(r, t)$  in the fluid and a function  $R(t)$ , where  $r=R(t)$  is the equation of the front. The system has the form

$$u_t + uu_r = \nu u_{rr} + \nu K(r^{-1}u_r - r^{-2}u),$$

$$T_t + uT_r = a(T_{rr} + Kr^{-1}T_r),$$

when  $r > R(t)$ ;  $u(\infty, t) = 0$ ,  $T(\infty, t) = T_\infty$ ,  $T(R, t) = T_s$ ,  $u(R, t) = -\varepsilon R'(t)$ ,  $T_r(R, t) = -\varepsilon R'(t)$ . Here  $K=0, 1, 2$  respectively for the plane, cylindrical or spherical front and the remaining undefined letters denote constants. {Reviewers remark: No initial conditions are prescribed. It seems doubtful that the above system would have a unique solution.} The author notes that the system can be satisfied by taking  $R=At^b$ ,  $T=g(rt^{-1})$  and  $u=t^{-1}/(rt^{-1})$ . He proceeds to find the constant  $A$  and the functions  $f$  and  $g$  and to discuss and make a graphical study of the resulting functions  $u$  and  $T$ . R. V. Churchill.

- Vodička, Václav. **Geschichteter Kreiszylinder im Felde periodischer Temperaturschwankungen.** Z. Angew. Math. Phys. 7 (1956), 422-427.

The author considers the propagation of concentric thermal, cylindrical waves through an infinitely long hollow cylinder composed of  $n$  co-axial layers. Each layer is assumed to have constant thermal properties which, however, differ from layer to layer. The inner and outer surfaces, i.e., the inner surface of cylinder 1, and the outer surface of cylinder  $n$ , are subjected to harmonically oscillating temperatures  $Be^{i\omega t}$  and  $Be^{i\omega t}$ , respectively. Intimate contact between the layers is not assumed and hence the radial temperature distribution is not continuous.

The problem is solved on classical lines by the super-

position of two solutions. Owing to the complexity of the problem, the procedure, rather than the final solution, is indicated. The final form brings out the physical significance of the solution in each layer. This can be regarded as the resultant of two cylindrical waves. *J. Kestin.*

**Dul'nev, G. N.; and Kondrat'ev, G. M. Generalized theory of the regular heat regime.** *Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk* 1956, no. 7, 71-85. (Russian)

The present paper contains a generalization of the authors' considerations published earlier [same *Izv.* 1955, no. 3, 130-138]. It analyzes the very important problem of the rate of temperature change of a body which contains sources or sinks of heat, as it approaches to an equilibrium state. The problem is treated in general terms on the assumption that the sources or sinks have constant strength, that the body is surrounded by a medium of constant temperature, that the thermal properties are constant, and that the coefficient of heat transfer on the surface of the body is constant.

The purely mathematical properties of the heat conduction equation made use of are well-known and imprecisely derived, but imaginatively applied to a general class of problems of engineering interest. The general solution  $u(x, y, z, t)$  of the heat conduction equation is represented as a difference  $t(x, y, z) - s(x, y, z, t)$  between the steady-state temperature  $t$ , and the time-dependent temperature  $s$ . For the present class of problems, the latter can be represented, as is known, in the form of a series of exponentials  $s = \sum_{j=0}^{\infty} A_j U_j e^{-m_j t}$ , where the  $U_j$ 's satisfy the Poisson equation  $\nabla^2 U_j + (m_j/a) U_j = 0$  and are time independent. The  $m_j$ 's form an ascending series, the smallest of them dominating after the lapse of an initial period of time. Thus at some instant the rate of heating or cooling at any point in the body is exponential, with the same exponent  $-m_0 \tau$ , and the plot  $\ln(t-s)$  versus time  $\tau$  is a straight line with slope  $-m_0$  throughout the body. When this occurs the body is said to have assumed its regular regime.

The value of  $m_0$  is shown to be independent of the power and distribution of the sources, and its relation to the coefficient of heat transfer at the surface is derived. The results are verified with the aid of simple experiments.

*J. Kestin* (Providence, R.I.).

**Kovtun, D. G. On certain series of the theory of heat conduction of Fourier-Poisson.** *Ukrain. Mat. Zh.* 8 (1956), 159-176. (Russian)

This paper is the continuation of an investigation of the expansions of functions in terms of solutions of the heat equation over a rod with homogeneous boundary conditions at the ends [same *Zh.* 7 (1955), 273-290; MR 17, 1079]. A contour integration method is used for establishing the expansions of arbitrary functions of bounded variation, and the theory is extended to the case of an infinite rod, when the series become integrals. The expansions are shown to be not unique. *J. L. B. Cooper.*

**Parkus, H. Periodisches Temperaturfeld im Keil.** *Österreich. Ing.-Arch.* 10 (1956), 241-243.

Paper contains solution for the Fourier equation of heat conduction which is valid inside a wedge of angle  $\alpha$ . One side of the wedge is at constant temperature whereas the other side has a temperature which fluctuates harmonically with time. Solution is obtained by separation in the usual manner. [The temperature  $T(r, \varphi, t)$  in polar coordinates is set equal to  $T = \theta e^{i\omega t} U(r, \varphi)$ ]. The resulting

equation (vibration equation) which involves spatial variables only,  $\nabla^2 U + k^2 U = 0$  is then solved subject to the boundary conditions  $U = \text{const}$  for  $\varphi = +0$ ,  $U = 0$  for  $\varphi = \alpha = 0$ , by the use of an integral transformation due to Kontorovich and Lebedev [*Acad. Sci. USSR. J. Phys.* 1 (1939), 229-241]. The solution is in terms of a definite integral which involves hyperbolic functions and a Bessel function of the third kind (Basset or Macdonald function). *J. Kestin* (Providence, R.I.).

**Chambers, L. G. A variational principle for the conduction of heat.** *Quart. J. Mech. Appl. Math.* 9 (1956), 234-235.

The author states a variational principle equivalent to the equation of heat conduction. The functional to be made stationary appears as an integral over the (spatial) region of heat flow, and the variation is performed with respect to the time rate of change of temperature, the temperature itself being unvaried. The second variation is positive definite, but the functional itself is not.

*R. Finn* (Pasadena, Calif.).

See also: Backus, p. 206; Boley, p. 216; Signorini, p. 246; Bishop, p. 250; Cohn, p. 276; Martynov, p. 314; Hemp, p. 348.

## Quantum Mechanics

**Prieto C., Fernando E. The field of Bhabha's equation.** *Univ. Nac. Autonoma Mexico. An. Inst. Fis.* 1 (1955), 95-100. (Spanish)

Bhabha has constructed an irreducible relativistic wave equation for a particle with two different mass states [*Phil. Mag.* (7) 43 (1952), 33-47; MR 13, 1010]. It is here shown that the manipulations with this equation are formally identical with those for the Dirac equation, as long as the explicit commutation relations of the matrices  $\alpha_\mu$  need not be used. The interaction of such particles with a scalar meson field is briefly considered.

*N. G. van Kampen* (Utrecht).

**Kamefuchi, Susumu; and Umezawa Hiroomi. On the renormalization theory of quantum electrodynamics.** *Progr. Theoret. Phys.* 15 (1956), 298-300.

This paper contains a brief discussion of the consistency of the renormalized quantum electrodynamics at high energies. *S. N. Gupta* (Detroit, Mich.).

**Rayski, J. On a bilocal interpretation of isotopic spin.** *Nuovo Cimento* (10) 3 (1956), 126-130.

In the author's theory of elementary particles [*Bull. Acad. Polon. Sci. Cl. III* 3 (1955), 255-257; MR 17, 114] the field variables are functions of the displacement operators — fourvectors  $x_\mu, d_\mu$ . These transform simultaneously under the transformations of the Lorentz group. They also transform under the unimodular group

$$x'_\mu = \alpha x_\mu + \beta d_\mu,$$

$$d'_\mu = \gamma x_\mu + \delta d_\mu,$$

which preserves the fourvector character and commutation relations. It is proposed that the field quantities form a basis in the vector space of an irreducible representation of the product of these groups, and that the transformation properties with respect to the unimodular group may be connected with isotopic spin. No attempt is

made to establish this connection explicitly.

*P. T. Matthews* (Birmingham).

**Watanabe, Satoshi.** Symmetry in time and Tanikawa's method of superquantization in regard to negative energy fields. *Progr. Theoret. Phys.* 15 (1956), 523-535.

The author describes a formalism of quantum field theory, in which he introduces "superquantization", "supercharge" and the "method of double inferential state-vector". He suggests that his formalism might be useful in the treatment of negative energy fields.

*S. N. Gupta* (Detroit, Mich.).

**Uhlhorn, U.** On the connection between transformations in classical mechanics and in quantum mechanics and the phase space representation of quantum mechanics. *Ark. Fys.* 11 (1956), 87-100.

The author shows the relation between the quantization prescriptions and the isomorphism of certain subgroups of the group of canonical transformations in classical mechanics with certain subgroups of the group of unitary transformations in quantum mechanics; also the connection of these with the phase space representation of quantum mechanics. The technique may prove helpful in quantum statistical mechanics.

*H. D. Block.*

**Smith, E. A.** Problems behind the wave equation. *Rev. Ci., Lima* 57 (1955), 88-93.

**Ter-Mikaelyan, M. L.** On the impact parameter method. *Akad. Nauk Armyan. SSR. Izv. Fiz. Mat. Estest. Tehn. Nauki* 9 (1956), no. 5, 77-90. (Russian. Armenian summary)

The author points out that the standard Weizsäcker-Williams method [see W. Heitler, *Quantum theory of radiation*, 3rd ed., Oxford, 1954, pp. 414-418] for the calculation of radiative processes of charged particles can be made to yield substantially more accurate results if the exact spectrum of equivalent transverse photons is used and the effective lower limit on the impact parameter is chosen as a suitable function of the energy transfer in the process in question. The application of this trick to the processes of bremsstrahlung with and without screening, and pair production is discussed. *A. S. Wightman.*

**Dolph, C. L.; and Ritt, R. K.** The Schwinger variational principles for one-dimensional quantum scattering. *Math. Z.* 65 (1956), 309-326.

From detailed mathematical discussion of one-dimensional, non-relativistic, quantum-mechanical scattering of a plane wave by a positive potential  $V(x)$ , it is concluded that if  $\int_{-\infty}^{\infty} V(x) dx < 2k$ , ( $k^2 = \text{energy}$ ), the Schwinger variational principle [Levine and Schwinger, *Phys. Rev.* (2) 74 (1948), 958-974; MR 10, 221] for determining the asymptotic behaviour of the transmitted wave is valid: if  $\infty > \int_{-\infty}^{\infty} V(x) dx \geq 2k$  an additional condition must be imposed. *C. Strachan* (Aberdeen).

**Stratonovič, R. L.** Gauge-invariant analogue of Wigner's distribution. *Dokl. Akad. Nauk SSSR (N.S.)* 109 (1956), 72-75. (Russian)

The author generalizes Wigner's expression for the quantum mechanical density in phase space [*Phys. Rev.* (2) 40 (1932), 749-759] to the case in which an external electromagnetic field is present. The final result is that

Wigner's formula

$$w(P, q) = (2\pi)^{-n} \int \exp[-i\tau P \rho(q + \frac{1}{2}\tau, g - \frac{1}{2}\tau) d\tau$$

holds in the presence of an electromagnetic field if  $P$  is interpreted as the gauge invariant quantity,  $p - eA$ , where  $p$  is the momentum and  $A$  the vector potential, provided that for  $\rho$  one uses the gauge invariant density matrix. The author also works out the equation of motion for  $w(P, q)$  and gives explicit formulae for the quantum corrections in the case of a relativistic system with no spin.

*A. S. Wightman* (Princeton, N.J.).

**Pease, Jane; and Pease, Robert L.** Intrinsic moments of elementary particles. *Phys. Rev.* (2) 104 (1956), 816-821.

The system of  $p$  particles (with different masses) being considered is assumed to be described by a field function  $\psi$  which satisfies

$$\left[ i\beta^\mu \frac{\partial}{\partial x^\mu} - A_\mu + k \right] \psi = 0$$

where  $A_\mu = (ie/\hbar c)A_\mu$ , and  $A_\mu$  represents the four potential of the external electromagnetic field, and  $\beta^\mu$  are matrices satisfying commutation rules given by Bhabha [*Rev. Mod. Phys.* 21 (1949), 451-462; MR 11, 764]. The authors develop a systematic method for extracting the intrinsic moments of a system with arbitrary  $n=2p$  or  $2p+1$ . Such moments are found to exist up to and including  $2^{n-1}$ -pole order. An explicit formula for the magnetic dipole moment is derived and compared to previously obtained results. *A. H. Taub* (Urbana, Ill.).

**Hack, M. N.** Multiple quantum transitions of a system of coupled angular momenta. *Phys. Rev.* (2) 104 (1956), 84-88.

The problem treated in this paper is that of a system of two coupled angular momenta, with given gyromagnetic ratios, interacting with each other and with an external magnetic field. The latter consist of a rotating radio frequency field of constant amplitude and frequency and a uniform static field of constant amplitude along the axis of rotation. Multi-quantum transitions of this system are investigated by time dependent perturbation theory in the uncertainty width region, and by actual integration of the Schrödinger equation for higher amplitudes of the radio frequency field. In the latter case a Rabi-type formula is derived. The paper claims to clarify the relationship between the steady state solution approach to this problem and the treatment first given by Mayer [*Naturwissenschaften* 17 (1929), 932]. *M. J. Moravcsik.*

**Klimontovič, Yu. L.** Determination of characteristic values of physical quantities by means of quantum distribution function. *Dokl. Akad. Nauk SSSR (N.S.)* 108 (1956), 1033-1036 (Russian)

Wigner's expression for the quantum mechanical density in phase space [*Phys. Rev.* (2) 40 (1932), 749-759] satisfies an equation of motion analogous to the classical equation of motion which equates the time derivative of a quantity to its Poisson bracket with the Hamiltonian. The author here proposes the use of that equation (instead of the Schrödinger equation for the wave function) for determining the energy proper values and other properties of bound states. He illustrates the method for the harmonic oscillator. *A. S. Wightman.*

**Faure, Robert.** Transformations isométriques en Mécanique analytique et en Mécanique ondulatoire. C. R. Acad. Sci. Paris 242 (1956), 2801-2803.

Considerato un integrale primo lineare delle equazioni canoniche della meccanica analitica, le condizioni che lo caratterizzano si identificano con quelle (di Killing) che esprimono la isometria del gruppo di trasformazioni nello spazio delle configurazioni, definito da un sistema differenziale del tipo

$$\frac{dq_i}{dt} = Q_i(q_1, \dots, q_n) \quad (i=1, \dots, n),$$

i valori generici delle  $q$  essendo i trasformati dei valori iniziali.

Se la trasformazione lascia invariato il potenziale, si può dedurre da essa un integrale primo lineare che vale sia in Meccanica analitica che in Meccanica ondulatoria, quando il sistema dinamico è tale che i suoi vincoli e il potenziale sono indipendenti dal tempo.

G. Lampariello (Roma).

**Baker, George Allen, Jr.** Degeneracy of the  $n$ -dimensional, isotropic, harmonic oscillator. Phys. Rev. (2) 103 (1956), 1119-1120.

The degeneracy of the energy levels of the  $n$ -dimensional isotropic harmonic oscillator is traced to the invariance of the Hamiltonian under the  $n$ -dimensional unitary group.

S. Deser (Copenhagen).

**Tolmačev, V. V.; and Tyablikov, S. V.** A method of calculating the partition functions for a ferromagnet with allowance for restrictions on the spin waves occupation numbers. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 1029-1031. (Russian)

The authors indicate briefly a method for calculating the sum of states of an ideal ferromagnet which is based on the notion of spin waves but avoids one of the customary approximations: the spin of each atom is not allowed to be arbitrarily large. The starting point is a definition of spin waves given by Holstein and Primakoff. The trick is to permit the occupation numbers of the spin states to run from zero to infinity but to annihilate the contributions from redundant states with a projection operator. A related trick was independently introduced by F.J. Dyson, but he used it in connection with a different definition of spin waves. The reader is referred to his papers [Phys. Rev. (2) 102 (1956), 1217-1230, 1230-1244; MR 17, 1165] for a fuller treatment, references, and a criticism of the Holstein-Primakoff definition of spin waves.

A. S. Wightman (Princeton, N.J.).

**Ouchi, Tadashi; Senba, Kei; and Yonezawa, Minoru.** Theory of mass reversal in the quantized field theory. Progr. Theoret. Phys. 15 (1956), 431-444.

Peaslee [Phys. Rev. (2) 91 (1953), 1447-1457] and Tiomno [Nuovo Cimento (10) 1 (1955), 226-232; MR 16, 1184] have considered a new field-theoretical transformation by which the mass constant changes sign and all spinors are transformed by the  $\gamma_5$ -matrix. This so-called mass reversal transformation is here studied for quantized fields. It is shown for spinor particles that it does not correspond to anything like a change in sign of the physical mass but that it affects the relative orientation of spin and momentum. The interactions invariant for mass reversal are discussed. This paper has led Watanabe to show [Progr. Theoret. Phys. 15 (1956), 81-83] that various other forms

of reversal can be constructed in the same line of thought as followed for mass reversal. L. Van Hove (Utrecht).

**Thellung, A.** On the energy spectrum in quantum hydrodynamics and the theory of helium II. Helv. Phys. Acta 29 (1956), 103-127.

The quantization of the equations of hydrodynamics was attempted by Kronig and Thellung [Physica 18 (1952), 749-761; MR 14, 427] in order to provide a basis for the theory of liquid helium II. Their analysis dealt only with irrotational motion. The extension to vortex motion was given by Ziman [Proc. Roy. Soc. London Ser. A. 219 (1953), 257-269] by the quantization of Clebsch's formulation of the hydrodynamic equations. The author continues this line of attack in an effort to calculate the spectrum of the Hamiltonian operator. Ordinary perturbation methods are used, and divergent sums and integrals are encountered which are reduced by a variety of cut-off procedures, as in other quantized field theories. The results obtained appear to have some non-trivial correlation with the empirical data despite the uncertainties of interpretation. On the mathematical side the theory suffers from the fundamental difficulties encountered in all quantized field theories in attempting to deal with continuous spectrum problems by perturbation methods. E. L. Hill (Minneapolis, Minn.).

**Soriano, S.** Perturbazione dei livelli energetici di una particella in una buca di potenziale sferoidale. Nuovo Cimento (10) 4 (1956), 657-660.

Methods previously given by Feenberg and Hammack [Phys. Rev. (2) 81 (1951), 285] and by Gallone and Salvetti [ibid. 82 (1951), 551] for computing the energy of a particle in a spheroidal potential well are taken to the next order of approximation in terms of the small parameters which prescribe the departure of the well from a spherical shape. The two methods are shown to lead to identical results in this order. The application of the result to a slightly deformed nucleus is discussed briefly.

H. C. Corben (Pittsburgh, Pa.).

**da Silveira, Adel.** On the theory of spin-two particles. Nuovo Cimento (10) 3 (1956), 513-516.

In a previous note [Phys. Rev. (2) 97 (1955), 1144] a theory of spin two particles was developed, without the introduction of an additional field, analogously to the Rarita-Schwinger theory of particles of spin three halves. For the case of zero mass the existence of a gauge transformation of the second kind is established in this paper.

P. T. Matthews (Birmingham).

**Taylor, J. C.** Renormalization in meson theories. Proc. Cambridge Philos. Soc. 52 (1956), 534-537.

Following the work of Gell-Mann and Low [Phys. Rev. (2) 95 (1954), 1300-1312; MR 16, 315], the author introduces a certain type of high-energy cut-off in the asymptotic propagation functions of the renormalized charge symmetrical pseudoscalar meson theory. It is then shown that when the cut-off energy tends to infinity, the so-called "bare-particle" coupling constant for the meson-nucleon interaction becomes imaginary.

S. N. Gupta (Detroit, Mich.).

**Iso, Chikashi; and Kawaguchi, Masaaki.** Note on the decay interactions of hyperons and heavy mesons. Progr. Theoret. Phys. 16 (1956), 177-188.

When decay-products can be assigned definite isotopic

spins, the authors assume that the decay Hamiltonian for an elementary particle transforms as a spinor in isotopic space. This hypothesis [also studies by G. Takeda, *Phys. Rev.* (2) **101** (1956), 1547-1551; R. Gatto, *Nuovo Cimento* (10) **3** (1956), 318-335; R. H. Dalitz, *Proc. Phys. Soc. Sect. A.* **69** (1956), 527-540] can explain the long-life of  $\theta^+$  particle compared to  $\theta^0$  and also gives information about branching-ratios of various decay-modes of  $\Lambda$  and  $\Sigma$  hyperons.  
A. Salam (Cambridge, Mass.).

**Shibata, Takashi.** Spin representation of the new fundamental group of transformations in relativistic quantum mechanics. *J. Sci. Hiroshima Univ. Ser. A.* **19** (1956), 491-498.

The author has previously defined a three parameter subgroup of the Lorentz group [same *J.* **16** (1953), 487-496; *MR* **15**, 752]. In this paper he determines the two component and four component spinor representation of this group.  
A. H. Taub (Urbana, Ill.).

### Relativity

**Reulos, René.** Nouvelle transformation en relativité restreinte. *Arch. Sci. Soc. Phys. Hist. Nat. Genève* **9** (1956), 114-117.

The author writes down a coordinate transformation, with complex coefficients, which preserves the metric in Minkowski space. He claims certain advantages for it, but does not suggest the physical interpretation which is needed for a transformation which takes real vectors into complex ones. The transformation matrices are just the "regular representation" of quaternions, and are well-known in that connection [cf. C. Lanczos, *Z. Physik* **57** (1929), 447-473; Blaton, *ibid.* **95** (1935), 337-354; see also Reulos, *Phys. Rev.* (2) **102** (1956), 535-536; *MR* **17**, 1142; and F. B. Estabrook, *ibid.* **103** (1956), 1579-1580].  
F. A. E. Pirani (London).

**Durand, Emile.** Définition d'un élément de longueur invariant et d'un tenseur antisymétrique qui généralise le vecteur unitaire de la tangente à une courbe, quand cette dernière est en mouvement. *C. R. Acad. Sci. Paris* **243** (1956), 570-572.

Reasons are given for modifying the classical definition of length element of a curve in the case of a curve in motion, and a new definition is based on an antisymmetric tensor  $\lambda^{pq}$ , called the unit tangent tensor of the curve. This tensor, which satisfies  $\frac{1}{2}\lambda_{pq}\lambda^{pq}=1$ , is defined by

$$i\lambda^{pq}=(\Delta l)^{-1}(\Delta x^p V^q - \Delta x^q V^p),$$

where  $V$  is the velocity 4-vector and  $\Delta l$  is the length element of the curve. Consequences of the definition are discussed, and applications are given to electromagnetic theory.  
A. G. Walker (Liverpool).

★ **Ginzburg, V. L.** On the use of artificial satellites of the earth to check the general theory of relativity. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.  
Translated from *Z. Eksp. Teor. Fiz.* **30** (1956), no. 1, 213-214.

**Bhattacharya, S.** On certain hydrodynamical considerations of an imperfect fluid in a general relativistic field. *Nuovo Cimento* (10) **4** (1956), 501-502.  
Condensations in a perfect fluid of general relativity

require, it is suggested, that the energy-tensor of the fluid be generalized to include terms involving viscosity. The condition for the conservation of mass is found and also a formula for the circulation along a closed curve. If this curve is a geodesic, the circulation is independent of time.

G. C. McVittie (Urbana, Ill.).

**Komar, Arthur.** Necessity of singularities in the solution of the field equations of general relativity. *Phys. Rev.* (2) **104** (1956), 544-546.

The author shows that if space-time is such that there exists a coordinate system in which

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta - (dx^4)^2 \quad (\alpha, \beta = 1, 2, 3)$$

and if various other hypotheses are satisfied, then either there is a finite value of  $x^4$  for which

$$\frac{\partial \log g}{\partial x^4}$$

is infinite or space-time is flat. This result is obtained as a consequence of the field equations without a cosmological term and the other reasonable hypotheses made. (It is remarked that it is not clear as to whether this result implies an essential singularity in space-time. The author states parenthetically: "... , no non-trivial non-singular solution of Eq. (1) (presumably  $R_{ij}=0$ ) has yet been found..." Such solutions are given in the paper by the reviewer [*Ann. of Math.* (2) **53** (1951), 472-490; *MR* **12**, 865] cited by the author.) A special case of the result of this paper is to be found in a paper by Raychaudhuri [*Phys. Rev.* (2) **98** (1955), 1123-1126; *MR* **16**, 1059].

A. H. Taub (Urbana, Ill.).

**Vescan, Teofil T.** Note sur l'interaction relativiste d'une charge ponctuelle et d'une sphère conductrice en rotation. *Com. Acad. R. P. Romine* **6** (1956), 259-261. (Romanian. Russian and French summaries)

The metric associated with the problem described in the title is

$$ds^2 = c^2 \left\{ 1 - \left[ \frac{Z}{r} + \frac{R}{2r^2} - \frac{R}{2(r^2 - R^2)} \right] \lambda^2 \right\} dt^2 - \left( 1 + \lambda \frac{Z}{r} + \frac{R\lambda}{2r^2} - \frac{R\lambda}{2(r^2 - R^2)} \right) (dx^2 + dy^2 + dz^2) + \frac{8ZqR^2\omega}{5mc^2 r^3} (x dy - y dx) dt,$$

and the force acting on the point charge is also determined.  
A. Erdélyi (Jerusalem).

**Subbotin, M. F.** Theory of relativity and celestial mechanics. *Astr. Zh.* **33** (1956), 251-258. (Russian. French summary)

An expository article written on the occasion of the 50th anniversary of the discovery of the theory of relativity.  
E. Leimanis (Vancouver, B.C.).

**Aymard, Alix.** Champs de tétrapodes. *C. R. Acad. Sci. Paris* **243** (1956), 885-888.

(I) Notation: Let  $u_\lambda$  be a set of (four) orthonormal unit vectors,  $u^\lambda$  the inverse set,  $g^{ab}$  the nonholonomic components of the metric tensor  $g^{\lambda\mu}$  with respect to the  $u$ 's so that

$$(1)a \quad \varphi^{ab} = 0,$$

where

$$(1)b \quad \varphi^{ab} \stackrel{\text{def}}{=} g^{\lambda\mu} u_{\lambda} u_{\mu} - g^{ab}.$$

Let  $L$  be a scalar function of  $u_{\lambda}$ ,  $u_{\lambda;\mu}$ ,

$$\mathcal{L} = -g^{\lambda\mu} L, \quad L_{,a} \stackrel{\text{def}}{=} \partial L / \partial u_a, \quad L_{,a}^{\lambda} = \partial L / \partial u_{\lambda;a},$$

and  $[\mathcal{L}]_a^{\nu}$  the Lagrange derivative of  $\mathcal{L}$ .  $K_{\lambda\mu\nu\omega}$  is the curvature tensor of  $g_{\lambda\mu}$  and  $K_{\mu\omega} = g^{\lambda\nu} K_{\lambda\mu\nu\omega}$ .

II). The variational principles applied to  $\mathcal{L}$  leads together with (1)b to the auxiliary Lagrange function  $\mathcal{L}' = \mathcal{L} + \lambda_{ab} \varphi^{ab}$  with Lagrange parameters  $\lambda_{ab} = \lambda_{(ab)}$ . Hence by virtue of (1)b

$$(2)a \quad [\mathcal{L}]_a^{\nu} = [\mathcal{L}']_a^{\nu} + 2\lambda_{ab} u_{,a}^b$$

so that the elimination of the  $\lambda$ 's leads to the field equations

$$(3) \quad [\mathcal{L}]_{a[\mu} u_{\lambda]} = 0.$$

A simple analysis shows

$$(4) \quad [\mathcal{L}]_a^{\nu} = \sqrt{(-g)} (L_{,a}^{\nu} - L_{,a}^{\lambda} u_{,\lambda}^{\nu})$$

so that by virtue of (2)a

$$(5) \quad 0 = [\mathcal{L}']_a^{\nu} u_{,\nu}^a = \sqrt{(-g)} \{ L_{,a}^{\nu} u_{,\nu}^a - L_{,a}^{\lambda} u_{,\lambda}^{\nu} u_{,\nu}^a \}.$$

On the other hand

$$(L_{,a}^{\nu} u_{,\nu}^a)_{;\lambda} = L_{,a}^{\nu} u_{,\nu;\lambda}^a + L_{,a}^{\lambda} u_{,\lambda;\nu}^a + L_{,a}^{\lambda} u_{,\lambda}^a K_{\lambda\mu\nu\omega}$$

so that (5) reduces to

$$(6) \quad (\delta_{\mu}^{\lambda} L - L_{,a}^{\nu} u_{,\nu}^a)_{;\lambda} = L_{,a}^{\nu} u_{,\nu}^a K_{\lambda\mu\nu\omega} g^{\omega b} u_{,b}^{\lambda}.$$

According to the author

$$(7) \quad L_{,a}^{\nu} u_{,\nu}^a K_{\lambda\mu\nu\omega} g^{\omega b} u_{,b}^{\lambda} = 0$$

and therefore the divergence of the momentum energy tensor  $\delta_{\mu}^{\lambda} L - L_{,a}^{\nu} u_{,\nu}^a$  vanishes. (Reviewer's remarks:

(1)  $g = -(\det(u_{\lambda}^a))^2$ . However (4) holds only if we consider  $g$  as independent of the  $u$ 's. (2) Let  $\tilde{V} = h^{\lambda\nu} u_{,\lambda;\nu}$ .  $2L \stackrel{\text{def}}{=} g_{ab} \tilde{V}^a \tilde{V}^b$ . Then

$$g^{ab} L_{,a}^{\nu} u_{,\nu}^a K_{\lambda\mu\nu\omega} = \tilde{V}_{,a}^a K_{\lambda\mu}^{\omega}$$

and, if  $\tilde{V}_{,a}^a \neq 0$ , this result contradicts (7). (3) The form of equations (4)–(7) depends on the choice of the  $u$ 's.)

V. Hlavatý (Bloomington, Ind.).

**Bergmann, Peter G. On Einstein's  $\lambda$  transformations.** Phys. Rev. (2) 103 (1956), 780–781.

The transformation of affine connections given by

$$\Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\alpha\beta}^{\gamma} + \delta_{\alpha}^{\gamma} \lambda_{\beta}$$

is called a " $\lambda$ -transformation." The invariance of the curvature with respect to  $\lambda$  transformations is shown to imply that the variation of a vector field parallel with respect to a curve produced by varying the curve but leaving the end points fixed is unaffected by a  $\lambda$  transformation. It is pointed out that similar results can be shown to hold for spinor fields. A. H. Taub (Urbana, Ill.).

**Horváth, J. I. Contribution to the unified theory of physical fields.** Nuovo Cimento (10) 4 (1956), 577–581.

Discussing Einstein's criteria of unified theory of physical fields it is suggested that the second criterion: "Neither the field equations nor the Hamiltonian function can be expressed as a sum of several invariant parts, but are formally unified entities" should be replaced by the weaker one: "Neither the field equations nor the Hamiltonian function can be expressed as the sum of several invariant parts which have an individual geometrical meaning" and Einstein's criteria should be completed by the following: "From the field equations of a (non-linear) unified field theory the correct equations of motion must be deduced." (Author's summary.)

V. Hlavatý (Bloomington, Ind.).

**Takasu, Tsurusaburo. Non-conjectural theory of relativity as a non-holonomic Laguerre geometry realized in the three-dimensional teleparallelismically torsioned Cartesian space fibered with non-holonomic actions.** Yokohama Math. J. 3 (1955), 1–52.

This is an attempt to relate the theory of relativity with some geometry of a Laguerre type. An excerpt appeared already in Proc. Japan Acad. 31 (1955), 606–609 [MR 17, 676]. There is a section on experimental data and a list of literature.

J. A. Schouten (Epe).

**\*Kratzer, Adolf. Relativitätstheorie. Ausarbeitungen mathematischer und physikalischer Vorlesungen. Bd. XVII. Aschendorffsche Verlagsbuchhandlung, Münster, 1956. vi+234 pp. DM 18.00.**

This is a well-written text-book covering "basic" relativity theory. Its merits are clarity and comparative brevity. Most of the essential material of the first fourteen chapters of Bergmann's "Introduction to relativity theory" [Prentice-Hall, New York, 1942; MR 4, 55] are covered in about half the space.

One third of the book is devoted to Special Theory including electrodynamics, one quarter to tensor analysis and the rest to Einstein's General Theory. The approach may be described as "simplified Weyl". The affine connection and metric are introduced quite separately, as is desirable. The three tests of Einstein's theory are discussed clearly and in detail, but the references to observation are rather out-of-date. In particular, the author continues the universal practice of the books on relativity of accepting St. John's opinion [Astrophys. J. 67 (1928), 195–239] that observations of spectral shift in the sun are not inconsistent with Einstein's theory, whereas recent opinion [e.g., M. G. Adam, Monthly Not. Roy. Astr. Soc. 112 (1952), 546–569] is that no known hypothesis can reconcile the observed solar spectral shifts with Einstein's prediction.

Even though he has not read a clearer presentation of Einstein's theory, this book left the reviewer with a sense of dissatisfaction. Perhaps the presentation is too neat. It suggests that Einstein's achievement must be accepted complete and uncritically. It is the prevalence of this attitude, perhaps, which accounts for the fact that except for the important work of Lichnerowicz and his school, nothing seems to have happened in relativity theory since 1920 [apart from 20 pages on cosmology everything in the present book was in Hermann Weyl's "Raum, Zeit, Materie", 4th ed., Springer, Berlin, 1921]. In the early days of relativity, when tensor analysis was regarded as impossibly difficult and abstruse, such an attitude was understandable. Now that tensors are the

common tools of any intelligent undergraduate, the time has perhaps arrived when a rethinking of the foundations of relativity can be attempted. Possibly the best starting-point for this would be A. N. Whitehead's "The principle of relativity" [Cambridge, 1922] to which the present author makes no reference. *A. J. Coleman* (Toronto, Ont.).

See also: Moffat, p. 332; Kolsrud, p. 347; Shibata, p. 362; Elias, p. 365.

### Astronomy

**Proskurin, V. F.** Theory of the motion of Ceres. Trudy Inst. Teoret. Astr. 2 (1952), 3-184. (Russian)

After a short history of the theories of motion of the first minor planet Ceres, discovered in 1801, the author points out the necessity of developing a satisfactory analytic theory of motion of this planetoid. The method used by him is essentially that of P. Hansen for determination of absolute perturbations of minor planets as modified by G. W. Hill. Chap. 1 gives a detailed exposition of Hill's method for determining perturbations of the first order with respect to the perturbing masses. Much attention is paid to the method of determination of the constants of integration. Chap. 2 is concerned with determination of Jupiter perturbation of the first order of Ceres. Chapters 3, 4 and 5 contain the Saturn, Uranus and Neptune perturbations, and the perturbations of the inner planets, all of the first order, respectively. Chap. 6 is concerned with a detailed analysis of perturbations, obtained by various methods and various authors prior to the present author's work, and with comparison of these results with those of the author. In Chap. 7 the secular perturbations of the elements of the orbit of Ceres are determined and a transition is made from the secular perturbations of the elements to those of the coordinates of Ceres. The method used for the determination of the secular perturbations is that of Gauss, modified by Hill, Halphen and Goryačev. In the final 8th Chapter the determination of constants of integration is carried out as outlined in Chap. 1 and the results on Ceres perturbations caused by the major planets are given in the form of certain tables.

This work which constitutes the first part of an analytic theory of motion of Ceres and which is devoted to the theory of absolute perturbations of Ceres, of the first order with respect to the perturbing masses, is supposed to be followed by a second part in which the determination of higher order perturbations will be given.

*E. Leimanis* (Vancouver, B.C.).

**Cimino, Massimo.** Sulla stabilità degli ammassi globulari nella più generale ipotesi della distribuzione sferica della loro densità. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 217-223.

The author examines the stability conditions for spherically symmetrical globular clusters gravitating in circular orbits around the galactic center. The necessary and sufficient condition for stability is that the Jacobi integral for a point at a distance  $r$  from the center of the cluster be equal to zero. Assuming an arbitrary density function for the cluster, such as to make the Newtonian potential  $U(r) > 0$ , it is shown that the total energy  $E$  of any given member of the cluster must then satisfy the two

conditions

$$(I) \quad 0 > E > -U(0) \quad (U(0) > 0),$$

$$(II) \quad 2[U(r) + E] = -3n^2 r^2,$$

where  $n$  is the angular velocity of the center of the cluster in its circular orbit. The author shows that the stability conditions found by Charlier and by Picart for the particular case of homogeneous clusters can be derived from the general conditions (I) and (II). *L. Jacchia.*

**Alekseev, V. M.** Exchange and capture in the three-body problem. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 599-602. (Russian)

Let three particles  $P_0, P_1, P_2$  subject to their mutual attractions be given. Denote by  $m_i$  the mass of  $P_i$  ( $i=0, 1, 2$ ), and by  $r_i, v_i$  the radius vector and the velocity of  $P_i$  respectively. Further, let  $r_{ik} = |r_{ik}| = |r_i - r_k|$ .

We say that a capture takes place between the particles  $P_0$  and  $P_1$  if for  $t \rightarrow -\infty$  all  $r_{ik} \rightarrow \infty$  while for  $t \rightarrow +\infty$  we have  $r_{01} < C$  ( $C$ -finite),  $r_{12}, r_{02} \rightarrow \infty$ . We say that an exchange takes place in the same system if for  $t \rightarrow +\infty$  we have  $r_{02} < C$ ,  $r_{12}, r_{02} \rightarrow \infty$  while for  $t \rightarrow -\infty$  we have  $r_{01} < C$ ,  $r_{12}, r_{02} \rightarrow \infty$ .

The author constructs examples of exchange and capture under the assumption that  $m_0 \gg m_1, m_0 \gg m_2$ . For reasons of simplicity it is assumed that  $m_0 = 1, m_1 = m_2 = m$ . The results obtained hold if  $a < m_1/m_2 < A$ , where  $a$  and  $A$  are certain constants.

The motion is considered in the 18 dimensional phase space of radii vectors and velocities of the particles, and it is shown that the sets of initial values leading to an exchange or a capture have positive measure (for positive as well as for negative values of the energy constant of the system). Contrary to L. Becker [Monthly Not. Roy. Astr. Soc. 80 (1920), 590-597] and O. Yu. Šmidt [Dokl. Akad. Nauk SSSR (N.S.) 58 (1947), 213-216] who both used the method of numerical integration in constructing examples of exchange and capture orbits respectively, the author uses only analytic methods. *E. Leimanis.*

**Agostinelli, Cataldo.** Piccoli movimenti in una massa gassosa stellare in evoluzione adiabatica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 212-217.

Starting from the general dynamic equation of a perfect fluid and the equation of state, the density  $\rho$  and the velocity  $v$  of gas particles can be expressed by a system of partial differential equations of the 2nd order coupled with the continuity equation. The author considers the case of small velocities, in which the terms in  $v$  of order higher than the first can be neglected; in the case of adiabatic equilibrium his equations reduce to Emden's equation.

The introduction of a velocity potential, function of position and time, leads to a system of 4th order partial differential equations; the potential function itself can be expanded in a series of spherical harmonics, for which the general equations are derived. Considering only sinusoidal solutions of a given frequency, time can be eliminated from these equations, which then reduce to ordinary differential equations of the 4th order with the distance from the center of the star,  $r$ , as independent variable. The case of purely radial oscillations leads to 2nd order partial differential equations, which again, for sinusoidal oscillation of a given frequency, reduce to a single ordinary differential equation of the 2nd order. *L. Jacchia.*

Havlicek, F. I. Über eine kosmologische Differentialgleichung. Conseil Acad. RPF Yougoslavie. Bull. Sci. 2 (1956), 97.

The author points out that if the Newtonian constant of gravitation is taken to depend on time in such a way that  $G \sim t^{-2}$ , then for a cosmological space with constant density and zero pressure the function  $R(t)$  satisfies a second order differential equation which can be transformed into the Bessel equation and which has solutions similar to the function occurring in the Jordan theory of gravitation.

A. H. Taub (Urbana, Ill.).

Elias, H. Vierdimensionale Geometrie und ihre praktische Anwendung zur Erklärung kosmologischer Probleme. Experientia 12 (1956), 362-364.

The author claims to give "physical reality" to 4-dimensional geometry, but no definition of this "reality" is provided. He asserts that the development of a spiral arm of a galaxy can be described by the, apparently time-changing, intersection of a "hyperbody" in 4-dimensions with 3-dimensional space. Neither the nature of the "hyperbody" nor the nature of the time-dependence are specified. It is not clear to this reviewer whether the 4-dimensional geometry envisaged has a metric and, if so, what its signature is.

G. C. McVittie (Urbana, Ill.).

Bracewell, R. N. Two-dimensional aerial smoothing in radio astronomy. Austral. J. Phys. 9 (1956), 297-314.

See also: Cohn, p. 283; Stone, p. 329; McCrea, p. 355; Subbotin, p. 362.

### Geophysics

Maximon, L. C., and Morgan, G. W. A theory of tidal mixing in a "vertically homogeneous" estuary. J. Marine Res. 14 (1955), 157-175.

Les auteurs étudient la distribution d'un solvant (sel) dans des estuaires influencés par les marées et le courant marin, les autres facteurs étant supposés négligeables.

Ils considèrent un canal composé de deux régions superposées, l'une de composition salinité  $s_1$ , et de vitesse  $u_1$ , et d'autre  $s_2$ , de vitesse  $u_2$ , en utilisant les équations de continuité pour les deux milieux et celle du total et en introduisant les équations aux limites correspondantes. En utilisant quelques hypothèses simplificatrices ils arrivent à calculer le flux de salinité. Enfin ils généralisent ses résultats.

M. Kiveliovitch (Paris).

Cuénod, M. Contribution à l'étude des crues. Détermination de la relation dynamique entre les précipitations et le débit des cours d'eau au moyen du calcul à l'aide de suites. Principe et application. Houille Blanche 11 (1956), 391-403, discussion 346-347.

The author establishes Volterra-type integral equations for the relation between rainfall and discharge. Empirical values are used for the kernel, obtained, in the present case, from observations made on the river Krummbach. (It seems to the present reviewer that a good deal of further work has to be done before the author's theory can be evaluated. For, in practice, the ensuing system of linear simultaneous equations might be of an ill-con-

ditioned type. Some of the difficulties can, however, be avoided, say, by introducing relatively accurate statistical distribution functions.)

K. Bhagwandin (Oslo).

Veronis, G., and Morgan, G. W. A study of the time-dependent wind-driven circulation in a homogeneous, rectangular ocean. Tellus 7 (1955), 232-242.

Ce problème a attiré dernièrement l'attention d'un très grand nombre de chercheurs surtout américains et japonais.

Les auteurs considèrent des équations Navier-Stokes d'un fluide visqueux rapporté à une sphère en rotation. Pour simplifier le problème on suppose: 1) que le fluide est incompressible, 2) les termes de l'accélération verticale et les termes provenant de la viscosité sont négligeables dans l'équation du mouvement vertical, 3) on néglige les termes non linéaires.

En introduisant toutes ces simplifications on arrive à un système d'équations transformé en un autre système sans dimensions. Pour pouvoir résoudre ces équations les auteurs appliquent la méthode des perturbations en développant les termes suivant un très petit paramètre qui est à un facteur près la fréquence de la variation du vent. On obtient certains résultats pour l'intensité des courants et des phases.

M. Kiveliovitch (Paris).

Adem, Julián. A series solution for the barotropic vorticity equation and its application in the study of atmospheric vortices. Tellus 8 (1956), 364-372.

Ogurcov, K. I. Quantitative investigation of wave processes in elastic half-space for various types of acting forces. Leningrad. Gos. Univ. Uč. Zap. 208. Ser. Mat. Nauk 30 (1956), 142-220. (Russian)

The author remarks that proposing both qualitative and quantitative analysis of solutions of problems in the theory of elasticity in the case of a half-space, he hopes to contribute to the proper formulation of the laws of moving seismographic waves. This will enable one to examine precisely the conditions for amplitudes and modes of earth's waves. In the first chapter the author gives general remarks concerning the character of elastic perturbations in a semi-space. Using a cylindrical polar coordinate system he writes down the formulas for the horizontal and vertical displacements due to acting force. These equations were derived in a previous work of himself and Petrašen' [same Zap. 149 (1951)]. In the second chapter he calculates the magnitudes of displacements on the axis of symmetry, on the boundary of the semi-space and on the frontal parts of waves of the following kinds: longitudinal, transverse and conical. In the third chapter Ogurcov calculates the modification of the field of displacements as a consequence of the elastic waves covering the semi space. In the fourth chapter he calculates the form and the intensity of the waves of various kinds for a few cases of acting force (horizontal force, vertical, etc.). The calculated examples are: displacements on the vertical axis passing through the point of action of the force; displacements on the radii from the origin at various angles to the axis of the semi-plane, etc. The results, depending strongly upon the kind of acting forces, are presented in the form of tables and diagrams. The contents of the paper is limited to the calculation of particular numerical examples, using the results of the theory derived in the paper mentioned above.

M. Z. v. Krzywoblocki (Urbana, Ill.).

## OTHER APPLICATIONS

*Games, Economics*

Kaysen, Carl. The minimax rule of the theory of games, and the choice of strategies under conditions of uncertainty. *Metroecon.* 4 (1952), 5-14.

Freund, Rudolf J. The introduction of risk into a programming model. *Econometrica* 24 (1956), 253-263.

Baldwin, Roger R.; Cantey, Wilbert E.; Maisel, Herbert; and McDermott, James P. The optimum strategy in blackjack. *J. Amer. Statist. Assoc.* 51 (1956), 429-439.

Simon, Herbert A. A comparison of game theory and learning theory. *Psychometrika* 21 (1956), 267-272.

Foster, Caxton; and Rapoport, Anatol. Parasitism and symbiosis in an  $N$ -person non-constant-sum continuous game. *Bull. Math. Biophys.* 18 (1956), 219-231.

★ Vogel, Walter. Eine allgemeine Klasse von Zwei-Personen-Spielen. Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 73-79. Deutscher Verlag der Wissenschaften, Berlin, 1956. Expository.

Brems, Hans. The foreign trade accelerator and the international transmission of growth. *Econometrica* 24 (1956), 223-238.

The model considered has two countries, each with two sectors, firms and households. Households buy only domestically, but firms may use both foreign and domestic producers' goods, the amount of each being proportional to output. All producers' goods are retired after  $L$  time units. Services proportional to output are also needed by firms and purchases only from domestic households. Households purchase domestic goods in an amount proportional to their real income. This model yields a system of difference equations, which is finally reduced to a pair of equations of order  $L$ .

A series of economically plausible values are given to the coefficients. In some cases (15 out of 36) the solution corresponding to the dominant root yields absurd results in that, for example, consumption will be negative. In the remaining cases a common equilibrium growth path for all magnitudes (domestic and foreign) is arrived at, corresponding to the dominant characteristic root.

K. J. Arrow (Stanford, Calif.).

Phipps, Cecil G. Money in the utility function. *Metroecon.* 4 (1952), 44-65.

The author sketches a general equilibrium model with firms maximizing profit subject to a production function, and individuals maximizing utility subject to a budget restraint. To justify the holding of money, the utility function must include as a variable in some form the stock of money held. The author suggests this be done by making the utility function depend on the following variables: the rates of consumption of commodities (the usual arguments); the amounts of each commodity which could be purchased with the stock of money  $M$ ; the

amounts of each commodity which could be purchased with the stock of invested savings  $S$ ; and the stocks of commodities held by the individual. K. J. Arrow.

Franckx, Ed. La théorie métamathématique du jury. I. Assoc. Actuar. Belges. Bull. no. 58 (1956), 11-23.

Imagine  $n$  jurists, set to weigh  $m$  statements, expressing their findings in sentences like "statement 3 is more credible than statement 5, which is just as credible as statement 2, which is ... more credible than statement 1," and imagine a decision function that reconciles these  $n$  sentences in a new sentence of the same kind: the verdict. The author claims that if the verdict is sensitive to the findings of the individual jurist and insensitive to rearrangements of jurists and of statements, then there is just one consistent decision function, two versions of which he exhibits. When  $m=2$ , the result is known, and the decision function favors the statement that the majority of jurists think most credible [see K. O. May, *Econometrica* 20 (1952), 680-684; 21 (1953), 172-173; MR 14, 392, 778]. H. P. McKean, Jr.

Arrow, Kenneth J. The determination of many-commodity preferences scales by two-commodity comparisons. *Metroecon.* 4 (1952), 105-115.

Consider a weak preference ordering defined over the nonnegative orthant of  $n$ -dimensional commodity space, representable by a continuous utility function, and such that the demand for no commodity is satiated. Call a "ration plane" any plane section of commodity space which is such that  $n-2$  of the coordinates are held fixed and only the two remaining commodity amounts are allowed to vary. Theorem: The preference relation between any pair of points can be determined via a chain of intermediate comparisons solely between pairs of points lying in successive ration planes. That is, if one has a way of determining the order of any pair of points which lie in some ration plane, then the ordering of any pair of points is determined. R. Solow (Cambridge, Mass.).

Uzawa, Hirofumi. Note on preference and axioms of choice. *Ann. Inst. Statist. Math.*, Tokyo 8 (1956), 35-40.

Let a choice function  $C(X)$  associate with each set in a class  $\mathfrak{B}$  a non-empty subset. If  $P$  is a preference relation, the derived choice function is,

$$C(X) = \{x^0; x^0 \in X \text{ and } xPx^0 \text{ for all } x \in X\}.$$

For any choice function, define  $xPy$  to mean that for some  $X$ ,  $x \in C(X)$ ,  $y \in X - C(X)$ , and define  $xP^*y$  to mean the existence of  $x^1, \dots, x^r$  such that  $xPx^1$ ,  $x^1Px^2$ , ...,  $x^rPy$  ( $i=1, \dots, r-1$ ),  $x^rPy$ . A choice function is said to be rational if for any  $x$  and  $y$ , if  $x, y \in C(X)$  for some  $X$  in  $\mathfrak{B}$ , then  $xP^*y$  [the so-called Strong Axiom of Revealed Preference; see H. Houthakker, *Economica*, (N.S.) 17 (1950), 159-174; MR 13, 146].

The principal results of the paper are to show the close relation between preference relations (derived from weak orderings) and rational choice functions. (1) If  $P$  is a preference relation and  $C(X)$  the derived choice function, then  $C$  is rational and  $xPy$  implies  $xP^*y$  (if  $B$  contains all finite sets, then the word, "implies", can be replaced by the word, "equivalent"). (2) If  $C$  is rational, then  $P^*$  is a

preference relation and  $C$  is the choice function derived from  $P^*$ . Some other similar results are obtained, particularly when the underlying space has a topological structure compatible in an obvious sense with the preference relation.  
K. J. Arrow (Stanford, Calif.).

**Patinkin, Don.** The limitations of Samuelson's "correspondence principle". *Metroecon.* 4 (1952), 37-43.

Consider an economy having  $n$  commodities whose prices are  $p_1, \dots, p_n$  and suppose that the excess demands are  $X_i(p_1, \dots, p_n, m)$  ( $i=1, 2, \dots, n$ ), where  $m$  is a parameter such as the bank rate. In dynamic theory the rates of the  $p_i$ 's are expressed in terms of the excess demands. Theorems about comparative statics are largely concerned with the signs of  $dp_i/dm$ , when the excess demands are all zero. Samuelson's "correspondence principle" [Foundations of economic analysis, Harvard, 1947, Ch. 9; MR 10, 555] is that in effect general theorems in comparative statics depend on restrictions derived from the assumption that the equilibrium position is stable, so that comparative statics depends on the dynamic theory. The author argues that the correspondence principle is insufficient for deriving the general theorems of comparative statics except perhaps in simple cases such as when  $dp_i/dt$  is proportional to  $X_i$ . The author appears to be questioning the usefulness of the correspondence principle, not its validity.  
I. J. Good (Cheltenham).

**Harsanyi, John C.** Approaches to the bargaining problem before and after the theory of games: a critical discussion of Zeuthen's, Hicks', and Nash's theories. *Econometrica* 24 (1956), 144-157.

**Brücker-Steinkuhl, K.** Stichprobenkarten mit Iterationen. *Mitteilungsbl. Math. Statist.* 8 (1956), 154-175.

**Baumol, William J.; and Ide, Edward A.** Variety in retailing. *Management Sci.* 3 (1956), 93-101.

**Percus, Jerome; and Quinto, Leon.** The application of linear programming to competitive bond bidding. *Econometrica* 24 (1956), 413-428.

**Bellman, Richard.** Mathematical aspects of scheduling theory. *J. Soc. Indust. Appl. Math.* 4 (1956), 168-205.  
An expository paper describing a number of representative problems, e.g. the Hitchcock-Koopmans transportation problem.

See also: Meinesz, p. 342.

### Biology and Sociology

**Rashevsky, N.** A neural mechanism for adjustment to optimal conditions, with possible reference to visual accommodation. *Bull. Math. Biophys.* 18 (1956), 189-198.

\* **Frishkopf, Lawrence S.** A probability approach to certain neuroelectric phenomena. *Tech. Rep.* 307, Massachusetts Institute of Technology, Research Laboratory of Electronics, Cambridge, Mass., 1956. iv+74 pp.  
A detailed discussion of measurements of electric

potential in the cochlea of an anesthetized cat, and the explanation of aspects of the observations in terms of a postulate of fluctuating neural thresholds. Further conclusions involve the rejection of a single-population hypothesis of neural units.  
E. Reich.

**Opatowski, I.; and Schmidt, George W.** Determination of diffusion and permeability coefficients in muscle. *Bull. Math. Biophys.* 14 (1952), 45-65.

A method attempting the determination of permeability coefficients between the cells of a tissue and the intercellular fluid, as well as the diffusion coefficient of the fluid. The authors attack the problem for an infinite plane strip, using an awkward construction to obtain a system of equations, differential in time and difference in space. From the asymptotic behavior of the solutions of the system the authors are able to obtain values of the coefficients from the experimental datum giving as a time function the total ion content of a tissue immersed in a fluid free of the given ion. {It is not clear to the reviewer that the approximations entailed by use of the difference method are any better than those made in using differential methods throughout. Furthermore, the appropriate differential equations are separable and may be treated by conventional methods. In the differential treatment of the problem the cylindrical case offers no greatly increased complexity and this case is certainly the closer to experiment.}  
A. A. Blank (Knoxville, Tenn.).

**Claringbold, P. J.** The within-animal bioassay with quantal response. *J. Roy. Statist. Soc. Ser. B.* 18 (1956), 133-137.

Illustrations are given of the increased accuracy obtained, in biological assays with quantal responses, by comparing treatments within, rather than between, animals. Satisfactory results are obtained by scoring responses as 0 or 1 and performing an analysis of variance.  
P. Armitage (London).

**Katz, Leo; and Wilson, Thurlow R.** The variance of the number of mutual choices in sociometry. *Psychometrika* 21 (1956), 299-304.

In a sociometric test of a group of  $N$  individuals, the pair  $i, j$  constitutes a dyad if they have chosen each other. In the case that each group member makes  $d$  independent choices at random from the other  $N-1$  members, the authors derive the variance of the resulting number of dyads. Next they derive the variance of the same quantity in the case that member  $i$  makes  $d_i$  choices. Finally, they give an expression for the frequency function of the number of dyads if each member makes but one choice.  
C. C. Craig (Ann Arbor, Mich.).

See also: Simon, p. 366.

### Information and Communication Theory

**Garner, W. R.; and McGill, William J.** The relation between information and variance analyses. *Psychometrika* 21 (1956), 219-228.

Shannon's measure of information, applied to analysis of variance situations, is shown to be capable of partition in an analogous way to the sum of squares. The interaction term is considered in some detail.  
D. V. Lindley.

- ★ **Zhitomirskii, V. I.** Determination of the probability of communication interference caused by interfering signals. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 11 pp. Translated from *Radiotekhnika* 10, No. 10, 1955, pp. 15-22.

**Sapožkov, M. A.** Correlational method of measurement of the coefficient of distortion of transmission. *Akust. Ž.* 2 (1956), 279-284. (Russian)

Elementary applications of correlation techniques to the study of noise and distortion in transmission; an analogue correlator is described. *S. K. Zarembo.*

**Schauffler, Rudolf.** Über die Bildung von Codewörtern. *Arch. Elek. Übertr.* 10 (1956), 303-314.

Linear formulas and corresponding charts are developed for the establishment of codes with code words having a uniform number of digits such that the common transmission errors such as interchange and replacement of digits can be both detected and corrected. Conditions are investigated under which two formulas are equivalent in the sense that they give rise to the same vocabularies.

*V. E. Beneš* (Murray Hill, N.J.).

See also: Kolmogorov, p. 324; Williams, p. 346.

### Control Systems

**Nedelcu, Mariana.** Considérations sur certains schémas à relais temporisés et à relais d'intensité. *Rev. Math. Pures Appl.* 1 (1956), no. 2, 199-217.

Boolean algebra is used to analyze and synthesize systems involving instantaneous, delay, and current controlled relays. *D. H. Lehmer* (Berkeley, Calif.).

**Moisil, Gr. C.; et Ioanin, Gh.** La synthèse des schémas à contacts et relais avec des conditions de travail données pour les éléments exécutifs. *Rev. Math. Pures Appl.* 1 (1956), no. 2, 167-198.

The authors apply Boolean algebraic methods to the

design of relay systems for eight different examples involving one or two pushbuttons and up to four lamps which are lit or not according to stated rules. Every detail is explained. *D. H. Lehmer* (Berkeley, Calif.).

★ **Fink, L. M.** Apropos 'The limiting capacity of a communication system' by A. A. Kharkevich and E. L. Blokh. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 3 pp. Translated from *Radiotekhnika* 10, no. 10, 1955, pp. 74-75.

**Tihonov, V. I.** Effect of small fluctuations on an electron relay. *Vestnik Moskov. Univ.* 11 (1956), no. 5, 31-41. (Russian)

The statistics of operation are analyzed of a vacuum tube, monostable, multi-vibrator, driven by a periodic signal  $u(t)$ , with a superimposed noise  $\xi(t)$ . The probability distributions of  $\xi(t)$  and  $\dot{\xi}(t)$  are assumed independent and Gaussian. Under the assumption that the noise perturbs the switching time  $t_0$  only by a small amount  $\tau_1$  and that the noise at different switching times is uncorrelated, an expression is found for the probability distribution  $p(\tau_1)$ . The computation of the mean square deviation  $\sigma_1^2$  of  $\tau_1$  as the second order moment of  $p(\tau_1)$  is not possible because the resulting integral diverges, a consequence of the failure of the approximations involved in  $p(\tau_1)$  at large values of  $\tau_1$ . An estimate of  $\sigma_1^2$  is obtained by replacing  $p(\tau_1)$  by a Gaussian distribution with the mean square deviation

$$\sigma_1^2 = \frac{1}{p(\tau_1)} \frac{d^2 p(\tau_1)}{d\tau_1^2} \Big|_{\tau_1=0}$$

with the result  $\sigma_1 = \sigma_\xi / \dot{u}(t_0)$ , where  $\sigma_\xi$  is the mean square deviation of  $\xi$ .

An estimate is also obtained of the duration of conduction of the normally-off tube as caused by shot noise and thermal noise appearing at the grid of the normally-on tube. Under assumed values, the noise is found to put a limit on the accuracy of division or multiplication of the frequency of a quartz standard oscillator. *H. A. Haus.*

### HISTORY, BIOGRAPHY

**Stamatis, Evangelos.** Über den euklidischen Satz, Kreise verhalten sich zu einander, wie die Quadrate über den Durchmesser. *Prakt. Akad. Athēnōn* 30 (1955), 410-414. (Greek. German summary)

The author gives an alternative proof, based on a passage in Archimedes for the second part of Euclid's proof. *S. H. Gould* (Providence, R.I.).

**Galli, Mario.** Sulle idee di Leibniz circa la legge di conservazione delle forze vive. *Boll. Un. Mat. Ital.* (3) 11 (1956), 445-456.

★ **Boyer, Carl B.** History of analytic geometry. *Scripta Mathematica*, New York, 1956. ix+291 pp. \$6.00.

"The present history covers only such parts of analytic geometry as might reasonably be included in an elementary general college course. Consequently developments of the last hundred years or so are largely omitted. The manuscript of this work was completed about half a dozen years ago, and major portions of it have appeared from time to time in *Scripta Mathematica*." (From the author's preface.) The chapter headings are: earliest

contributions; Alexandrian age; medieval period; early modern prelude; Fermat and Descartes; age of commentaries; from Newton to Euler; definitive formulation; golden age. There is an analytical bibliography and an index.

**Nunziante-Cesàro, Carlo.** La risolvibile di Cartesio dell'equazione di quarto grado. *Period. Mat.* (4) 34 (1956), 169-170.

**Siebel, A.** Eine Herleitung der Simpsonschen  $\pi$ -Näherung nach Gregory. *Math.-Phys. Semesterber.* 5 (1956), 143-146.

**Krafft, Maximilian.** Elementare Bestimmung des Wertes des Wahrscheinlichkeitsintegrals. *Math.-Phys. Semesterber.* 5 (1956), 120-122.

An elementary proof hitherto unnoticed of the equation  $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$ , to be found in its essentials in the *Traité du calcul différentiel et intégral* of S. F. Lacroix (Paris, 1819).

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